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Quantum reduction and representation theory of superconformal algebras

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Abstract

We study the structure and representations of a family of vertex algebras obtained from affine superalgebras by quantum reduction. As an application, we obtain in a unified way free field realizations and determinant formulas for all superconformal algebras.

0. Introduction

In this paper we study the structure and representations of the vertex algebras $W_k(\mathfrak{g}, x, f)$ introduced in [KRW]. Let us briefly recall the construction of these vertex algebras. The datum we begin with is a quadruple $(\mathfrak{g}, x, f, k)$, where $\mathfrak{g}$ is a simple finite-dimensional Lie superalgebra with a non-zero even invariant supersymmetric bilinear form $(.\,|\,.)$, x is an ad-diagonalizable element of $\mathfrak{g}$ with eigenvalues in $\frac{1}{2}\mathbb{Z}$, f is an even element of $\mathfrak{g}$ such that $[x, f] = -f$ and the eigenvalues of ad x on the centralizer $\mathfrak{g}^f$ of f in $\mathfrak{g}$ are non-positive, and $k \in \mathbb{C}$. Recall that a bilinear form $(.\,|\,.)$ on $\mathfrak{g}$ is called even if $(\mathfrak{g}_{\bar 0}|\mathfrak{g}_{\bar 1}) = 0$, supersymmetric if $(.\,|\,.)$ is symmetric (resp. skewsymmetric) on $\mathfrak{g}_{\bar 0}$ (resp. $\mathfrak{g}_{\bar 1}$), invariant if $([a, b]|c) = (a|[b, c])$ for

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all $a, b, c \in \mathfrak{g}$. Note also that a pair (x, f) satisfying the above properties can be obtained by taking a non-zero nilpotent element $f \in \mathfrak{g}_0$ and including it in an $s\ell_2$-triple $\{e, x, f\}$, so that $[x, e] = e$, $[x, f] = -f$, $[e, f] = x$ (then, up to conjugacy, x is determined by f).

We associate to the quadruple $(\mathfrak{g}, x, f, k)$ a homology complex

$$\mathscr{C}(\mathfrak{g}, x, f, k) = (V_k(\mathfrak{g}) \otimes F^{\mathrm{ch}} \otimes F^{\mathrm{ne}}, d_0), \tag{0.1}$$

where $V_k(\mathfrak{g})$ is the universal affine vertex algebra of level k associated to $\mathfrak{g}$, F^{ch} is the vertex algebra of free charged superfermions based on $\mathfrak{g}_+ + \mathfrak{g}_+^*$ with reversed parity, F^{ne} is the vertex algebra of free neutral superfermions based on $\mathfrak{g}_{1/2}$ (see [K4] for definitions), and d_0 is an explicitly constructed odd derivation of the vertex algebra $\mathscr{C}(\mathfrak{g}, x, f, k)$ whose square is 0. Here $\mathfrak{g}_+$ (resp. $\mathfrak{g}_{1/2}$) denote the sum of eigenspaces with positive eigenvalues (resp. eigenspace with eigenvalue $1/2$) of $\mathrm{ad}\, x$ (see Section 1). The vertex algebra $W_k(\mathfrak{g}, x, f)$ is the homology of the complex (0.1). In the case when the pair (x, f) can be included in an $s\ell_2$-triple, we denote this vertex algebra by $W_k(\mathfrak{g}, f)$.

Our main result on the structure of the vertex algebras $W_k(\mathfrak{g}, x, f)$ is Theorem 4.1 which states that the jth homology of the complex (0.1) is zero if $j \neq 0$ and the 0th homology vertex algebra (which is $W_k(\mathfrak{g}, x, f)$) is strongly generated by fields $J^{\{a_i\}}$, where $a_1, \ldots, a_s$ is a basis of $\mathfrak{g}^f$ consisting of eigenvectors of $\mathrm{ad}\, x$. This theorem also gives a rather explicit form of these fields. Furthermore, provided that $k \neq -h^\vee$, we have an explicitly constructed energy-momentum field $L(z)$ (see (2.2)), with respect to which the fields $J^{\{a_i\}}$ have conformal weight $1 - m_i$, where $[x, a_i] = m_i a_i$. Theorem 2.1 of the present paper gives explicit formulas for the fields $J^{\{a_i\}}$ in the cases when $m_i = 0$ and $-1/2$, and the commutation relations between these fields.

We study in more detail the simplest, but a very interesting subclass of vertex algebras in question, namely $W_k(\mathfrak{g}, e_{-\theta})$ that correspond to $f = e_{-\theta}$, the lowest root vector of $\mathfrak{g}$ (for some choice of positive roots), see Theorem 5.1. These vertex algebras are characterized among all vertex algebras $W_k(\mathfrak{g}, x, f)$ by the property that they are strongly generated by the energy-momentum field and the fields of conformal weight 1 and 3/2. Moreover, it turns out that, under some natural assumptions, any vertex algebra satisfying this property is one of the vertex algebras $W_k(\mathfrak{g}, e_{-\theta})$ [D,FL]. Thus, all well-known superconformal algebras are covered by this construction:

$W_k(s\ell_2, e_{-\theta})$ is the Virasoro vertex algebra,
$W_k(s\ell_3, e_{-\theta})$ is the Bershadsky–Polyakov algebra [B],
$W_k(spo(2|1), e_{-\theta})$ is the Neveu–Schwarz algebra,
$W_k(spo(2|m), e_{-\theta})$ for $m \geqslant 3$ are the Bershadsky–Knizhnik algebras [BeK],
$W_k(s\ell(2|1) = spo(2|2), e_{-\theta})$ is the $N = 2$ superconformal algebra,
$W_k(s\ell(2|2)/\mathbb{C}I, e_{-\theta})$ is the $N = 4$ superconformal algebra,
$W_k(spo(2|3), e_{-\theta})$ tensored with one fermion is the $N = 3$ superconformal algebra (cf. [GS]),

$W_k(D(2,1;a), e_{-\theta})$ tensored with four fermions and one boson is the big $N=4$ superconformal algebra (cf. [GS]).

Theorem 5.1 also provides a construction of the vertex algebra $W_k(\mathfrak{g}, x, f)$ as a subalgebra of $V_{\alpha_k}(\mathfrak{g}_{\leqslant}) \otimes F^{\mathrm{ne}}$, where $\mathfrak{g}_{\leqslant}$ is the sum of eigenspaces of $\operatorname{ad} x$ with non-positive eigenvalues and α_k is a "shifted" 2-cocycle on $\mathfrak{g}_{\leqslant}[t, t^{-1}]$. Therefore the homomorphism $\mathfrak{g}_{\leqslant} \to \mathfrak{g}_0$, where $\mathfrak{g}_0$ is the centralizer of x, induces a realization of $W_k(\mathfrak{g}, x, f)$ inside $V_{\alpha_k}(\mathfrak{g}_0) \otimes F^{\mathrm{ne}}$. In particular, this construction gives an explicit free field realization of all vertex algebras $W_k(\mathfrak{g}, e_{-\theta})$ (Theorem 5.2), recovering thereby all previously known free field realizations of superconformal algebras (see [K4] and references there).

Given a highest weight module P of level k over the affine Lie superalgebra $\hat{\mathfrak{g}}$ associated to $\mathfrak{g}$, we have the associated $\mathscr{C}(\mathfrak{g}, x, f, k)$-module $\mathscr{C}(P) = P \otimes F^{\mathrm{ch}} \otimes F^{\mathrm{ne}}$ with the differential d_0^P. The homology $H(P)$ of the complex $(\mathscr{C}(P), d_0^P)$ is a $W_k(\mathfrak{g}, x, f)$-module. We show that if P is a generalized Verma module over $\hat{\mathfrak{g}}$, then $H_j(P) = 0$ for $j \neq 0$ (Theorem 6.2), and that $H_0(P)$ is a Verma module over $W_k(\mathfrak{g}, x, f)$ if P is a Verma module over $\hat{\mathfrak{g}}$ (Theorem 6.3). Since the singular weights of a Verma module over $\hat{\mathfrak{g}}$ are known [K2], this allows us to construct sufficiently many singular weights for a Verma module over $W_k(\mathfrak{g}, x, f)$. As a result, we get an explicit determinant formula for all vertex algebras $W_k(\mathfrak{g}, e_{-\theta})$ (Theorem 7.2), which includes as special cases all previously known determinant formulas for superconformal algebras (in the "untwisted" sector; the twisted sector will be treated in [KW5]).

The special cases of free field realization and determinant formulas for the Virasoro, $N = 1, 2, 3, 4$ and big $N = 4$ superconformal algebras are considered in Section 8. (Due to [FK] these are all finite simple formal distribution Lie superalgebras which admit a central extension containing a Virasoro subalgebra with a non-trivial center.)

In this paper we use some very simple homological arguments. The necessary general facts about homology are collected in Section 3.

Our paper represents further development of the ideas of Feigin and Frenkel [FF1,FF2,FB] who considered the very important case of the principal nilpotent element f, and of the paper [BT], which treated the case when all eigenvalues of $\operatorname{ad} x$ are integers. In these works the homology of the corresponding BRST complex is studied and many results of the present paper are proved in the respective special cases, using similar methods. After the present paper was send to the archive, we learned about the paper [ST] where a BRST complex similar to ours is studied along similar lines, and the paper [BG] where free field realizations similar to ours are established.

In a forthcoming paper [KW6] we shall develop the theory of characters of the vertex algebras $W_k(\mathfrak{g}, e_{-\theta})$, building on [KW1,KW2,KW3,KW4,KRW].

Throughout the paper all vector spaces, algebras and tensor products are considered over the field of complex numbers $\mathbb{C}$, unless otherwise stated. We denote by $\mathbb{Z}$, $\mathbb{Z}_+$ and $\mathbb{N}$ the sets of all integers, all non-negative integers and all positive integers, respectively.

1. The complex $\mathscr{C}(\mathfrak{g},x,f,k)$ and the associated vertex algebra $W_k(\mathfrak{g},x,f)$

We recall here, following [KRW] (see also [K5]), the construction of a vertex algebra $W_k(\mathfrak{g},x,f)$ depending on a complex parameter k, via a differential complex $\mathscr{C}(\mathfrak{g},x,f,k)$, associated to a simple finite-dimensional Lie superalgebra $\mathfrak{g}$ with a non-degenerate even supersymmetric invariant bilinear form $(.|.)$, and a pair x,f of even elements of $\mathfrak{g}$ such that $\operatorname{ad} x$ is diagonalizable on $\mathfrak{g}$ with half-integer eigenvalues and $[x,f]=-f$.

We have the eigenspace decomposition of $\mathfrak{g}$ with respect to $\operatorname{ad} x$:

$$\mathfrak{g}=\bigoplus_{j\in\frac{1}{2}\mathbb{Z}}\mathfrak{g}_j, \tag{1.1}$$

and we let:

$$\mathfrak{g}_+=\bigoplus_{j>0}\mathfrak{g}_j,\quad \mathfrak{g}_-=\bigoplus_{j<0}\mathfrak{g}_j,\quad \mathfrak{g}_\leqslant=\mathfrak{g}_0\oplus\mathfrak{g}_-. \tag{1.2}$$

The element f defines a skew-supersymmetric even bilinear form $\langle\,.\,|\,.\,\rangle_{\mathrm{ne}}$ on $\mathfrak{g}_{1/2}$ by the formula:

$$\langle a,b\rangle_{\mathrm{ne}}=(f\,|[a,b]). \tag{1.3}$$

This bilinear form is non-degenerate if and only if the map

$$\operatorname{ad} f:\mathfrak{g}_{1/2}\to\mathfrak{g}_{-1/2}\ \text{ is an isomorphism}, \tag{1.4}$$

which we shall assume. Denote by $\mathfrak{g}^\natural$ the centralizer of f in $\mathfrak{g}_0$. By (1.4) the $\mathfrak{g}^\natural$-modules $\mathfrak{g}_{1/2}$ and $\mathfrak{g}_{-1/2}$ are isomorphic. The bilinear form $\langle\,.\,,\,.\,\rangle_{\mathrm{ne}}$ is invariant with respect to the representation of $\mathfrak{g}^\natural$ on $\mathfrak{g}_{1/2}$:

$$\langle[v,a],b\rangle_{\mathrm{ne}}+(-1)^{p(v)p(a)}\langle a,[v,b]\rangle_{\mathrm{ne}}=0\quad (v\in\mathfrak{g}^\natural,a,b\in\mathfrak{g}_{1/2}). \tag{1.5}$$

Denote by A_{ne} the vector superspace $\mathfrak{g}_{1/2}$ with the bilinear form (1.3). Denote by A (resp. A^*) the vector superspace $\mathfrak{g}_+$ (resp. $\mathfrak{g}_+^*$) with the reversed parity, let $A_{\mathrm{ch}}=A\oplus A^*$ and define an even skew-supersymmetric non-degenerate bilinear form $\langle\,.\,,\,.\,\rangle_{\mathrm{ch}}$ on A_{ch} by

$$\begin{aligned}&\langle A,A\rangle_{\mathrm{ch}}=0=\langle A^*,A^*\rangle_{\mathrm{ch}},\\ &\langle a,b^*\rangle_{\mathrm{ch}}=-(-1)^{p(a)p(b^*)}\langle b^*,a\rangle_{\mathrm{ch}}=b^*(a)\quad\text{for } a\in A,b^*\in A^*.\end{aligned} \tag{1.6}$$

Here and further, $p(a)$ stands for the parity of an (homogeneous) element of a vector superspace.

Following [KRW], introduce the differential complex $(\mathscr{C}(\mathfrak{g},x,f,k),d_0)$, where $\mathscr{C}(\mathfrak{g},x,f,k)$ is a vertex algebra depending on a complex parameter k and d_0 is an odd derivation of all products of this vertex algebra, such that $d_0^2 = 0$. We have:

$$\mathscr{C}(\mathfrak{g}, x, f, k) = V_k(\mathfrak{g}) \otimes F(\mathfrak{g}, x, f), \tag{1.7}$$

where $V_k(\mathfrak{g})$ is the *universal affine vertex algebra of level k* associated to $\mathfrak{g}$ and

$$F(\mathfrak{g}, x, f) = F(A_{\mathrm{ch}}) \otimes F(A_{\mathrm{ne}}), \tag{1.8}$$

where $F(A_{\mathrm{ch}})$ (resp. $F(A_{\mathrm{ne}})$) is the *vertex algebra of charged* (resp. *neutral*) *free superfermions* based on A_{ch} (resp. A_{ne}).

Recall that the vertex algebra $V_k(\mathfrak{g})$ is constructed using the *Kac–Moody affinization* of $\mathfrak{g}$ (cf. [K3]), which is the Lie superalgebra $\hat{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}K \oplus \mathbb{C}D$ with the commutation relations $(a, b \in \mathfrak{g}, m, n \in \mathbb{Z})$:

$$[at^m, bt^n] = [a, b]t^{m+n} + m\delta_{m,-n}(a|b)K, [D, at^m] = mat^m, [K, \hat{\mathfrak{g}}] = 0.$$

Let $\hat{\mathfrak{g}}' = \mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}K$ be its derived algebra. Then $V_k(\mathfrak{g}) = U(\hat{\mathfrak{g}}')/U(\hat{\mathfrak{g}}')(\mathfrak{g}[t] \oplus \mathbb{C}(K - k))$, as a left $\hat{\mathfrak{g}}'$-module. Here and further, $U(\mathfrak{p})$ stands for the universal enveloping algebra of the Lie superalgebra $\mathfrak{p}$. Introduce the *current* attached to $a \in \mathfrak{g}$: $a(z) = \sum_{n \in \mathbb{Z}} (at^n)z^{-n-1}$. Recall that the λ-bracket of the currents is $[a_\lambda b] = [a, b] + \lambda(a|b)k,\ a, b \in \mathfrak{g}$. Then, by the existence theorem [K4], the space $V_k(\mathfrak{g})$ carries a unique vertex algebra structure such that the vacuum vector $|0\rangle = 1$, the infinitesimal translation operator $T = -\frac{d}{dt}$, and $Y(at^{-1}|0\rangle, z) = a(z)$, $a \in \mathfrak{g}$.

Given a vector superspace A with an even skew-supersymmetric non-degenerate bilinear form $\langle . , . \rangle$, the associated vertex algebra $F(A)$ of free superfermions is defined by making use of the *Clifford affinization* of A, which is the Lie superalgebra $\hat{A} = A[t, t^{-1}] + \mathbb{C}K$ with the commutation relations $(a, b \in A, m, n \in \mathbb{Z})$:

$$[at^m, bt^n] = \delta_{m,-n-1}\langle a, b\rangle K, [K, \hat{A}] = 0.$$

Then $F(A) = U(\hat{A})/U(\hat{A})(A[t] \oplus \mathbb{C}(K - 1))$, as a left $\hat{A}$-module. The free superfermion attached to $\varphi \in A$ is $\varphi(z) = \sum_{n \in \mathbb{Z}} (\varphi t^n)z^{-n-1}$, and the λ-bracket is: $[\varphi_\lambda \psi] = \langle \varphi, \psi \rangle, \varphi, \psi \in A$. Likewise, $F(A)$ carries a unique vertex algebra structure with $|0\rangle = 1$, $T = -\frac{d}{dt}$ and $Y(\varphi t^{-1}|0\rangle, z) = \varphi(z)$, $\varphi \in A$.

The vertex algebra $F(A_{\mathrm{ch}})$ has the charge decomposition:

$$F(A_{\mathrm{ch}}) = \bigoplus_{m \in \mathbb{Z}} F_m(A_{\mathrm{ch}}),\ \text{where charge}\, \varphi(z) = -\text{charge}\, \varphi^*(z) = 1 \ \text{for}\ \varphi \in A, \varphi^* \in A^*.$$

Letting charge $V_k(\mathfrak{g}) = 0$, charge $F(A_{\mathrm{ne}}) = 0$, this induces the charge decompositions:

$$F(\mathfrak{g}, x, f) = \bigoplus_{m \in \mathbb{Z}} F_m, \quad \mathscr{C}(\mathfrak{g}, x, f, k) = \bigoplus_{m \in \mathbb{Z}} \mathscr{C}_m. \tag{1.9}$$

In order to define the differential d_0, choose a basis $\{u_\alpha\}_{\alpha \in S_j}$ of each $\mathfrak{g}_j$ in (1.1), and let $S = \coprod_{j \in \frac{1}{2}\mathbb{Z}} S_j$, $S_+ = \coprod_{j>0} S_j$. Let $p(\alpha) \in \mathbb{Z}/2\mathbb{Z}$ denote the parity of u_α, and let

$m_\alpha = j$ if $\alpha \in S_j$. Define the structure constants $c^\gamma_{\alpha\beta}$ by $[u_\alpha, u_\beta] = \sum_\gamma c^\gamma_{\alpha\beta} u_\gamma$ $(\alpha, \beta, \gamma \in S)$. Denote by $\{\varphi_\alpha\}_{\alpha \in S_+}$ the corresponding basis of A and by $\{\varphi^\alpha\}_{\alpha\in S_+}$ the basis of A^* such that $\langle \varphi_\alpha, \varphi^\beta \rangle_{\mathrm{ch}} = \delta_{\alpha\beta}$. Denote by $\{\Phi_\alpha\}_{\alpha\in S_+}$ the corresponding basis of A_{ne}, and by $\{\Phi^\alpha\}_{\alpha\in S_{1/2}}$ the dual basis with respect to $\langle\,.\,,\,.\,\rangle_{\mathrm{ne}}$, i.e., $\langle \Phi_\alpha, \Phi^\beta\rangle_{\mathrm{ne}} = \delta_{\alpha\beta}$. It will also be convenient to define Φ_u for any $u \in \sum_{\alpha\in S} c_\alpha u_\alpha \in \mathfrak{g}$ by letting $\Phi_u = \sum_{\alpha \in S_{1/2}} c_\alpha \Phi_\alpha$. Following [KRW], introduce the following odd field of the vertex algebra $\mathscr{C}(\mathfrak{g}, x, f, k)$:

$$\begin{aligned} d(z) = &\sum_{\alpha\in S_+} (-1)^{p(\alpha)} u_\alpha(z) \otimes \varphi^\alpha(z) \otimes 1 \\ &- \frac{1}{2} \sum_{\alpha,\beta,\gamma\in S_+} (-1)^{p(\alpha)p(\gamma)} c^\gamma_{\alpha\beta} \otimes \varphi_\gamma(z)\varphi^\alpha(z)\varphi^\beta(z) \otimes 1 \\ &+ \sum_{\alpha\in S_+} (f\,|u_\alpha) \otimes \varphi^\alpha(z) \otimes 1 + \sum_{\alpha\in S_{1/2}} 1 \otimes \varphi^\alpha(z) \otimes \Phi_\alpha(z). \end{aligned}$$

Then $d_0 \coloneqq \mathrm{Res}_z\, d(z)$ is an odd derivation of all products of the vertex algebra $\mathscr{C}(\mathfrak{g}, x, f, k)$ and $d_0^2 = 0$, since by [KRW], Theorem 2.1, $[d(z), d(w)] = 0$. Also, $d_0(\mathscr{C}_m) \subset \mathscr{C}_{m-1}$. Thus, $(\mathscr{C}(\mathfrak{g}, x, f, k), d_0)$ is a $\mathbb{Z}$-graded homology complex. The homology of this complex is a vertex algebra, denoted by $W_k(\mathfrak{g}, x, f)$, and called the *quantum reduction* for the triple $(\mathfrak{g}, x, f)$.

We shall often drop the tensor product sign $\otimes$, and also shall often drop z (for example we write d in place of $d(z)$, ∂a in place of $\frac{d}{dz} a(z)$, etc.) if no confusion may arise.

The most interesting pair x, f satisfying (1.4) comes from an $s\ell_2$-triple $\{e, x, f\}$, where $[x, e] = e$, $[x, f] = -f$, $[e, f] = x$. In this case we have a stronger property: $\mathfrak{g}^f \coloneqq \{a \in \mathfrak{g} \mid [f, a] = 0\} \subset \mathfrak{g}_{\leqslant}$, or, equivalently,

$$\mathfrak{g}^f = \bigoplus_{j\leqslant 0} \mathfrak{g}^f_j, \quad \text{where } \mathfrak{g}^f_j = \{a \in \mathfrak{g}_j \mid [f, a] = 0\}. \tag{1.10}$$

This property is immediate by the $s\ell_2$-representation theory. Since a nilpotent even element f determines uniquely (up to conjugation) the element x of an $s\ell_2$-triple (by a theorem by Dynkin), we use in this case the notation $C(\mathfrak{g}, f, k)$ for the complex and $W_k(\mathfrak{g}, f)$ for the quantum reduction. In fact, up to isomorphism, the vertex algebra $W_k(\mathfrak{g}, f)$ depends obviously only on the adjoint orbit of f in the even part of $\mathfrak{g}$. Note also that in this case the subalgebra $\mathfrak{g}^f_0$ coincides with the centralizer of $\{e, x, f\}$ in $\mathfrak{g}$.

Property (1.10) of the pair (x, f) will play an important role in the sequel. Note that this property is equivalent to

$$[f, \mathfrak{g}_j] = \mathfrak{g}_{j-1} \quad \text{if } j \leqslant \tfrac{1}{2}. \tag{1.11}$$

Indeed, $[f, \mathfrak{g}_j] \neq \mathfrak{g}_{j-1}$ for $j \leqslant \frac{1}{2}$ is equivalent to existence of a non-zero $a \in \mathfrak{g}_{-j+1}$ such that $([f, \mathfrak{g}_j]|a) = 0$, i.e., $([f, a]|\mathfrak{g}_j) = 0$, which is equivalent to $a \in \mathfrak{g}^f_{-j+1}$.

The pair (x,f) and the corresponding $\frac{1}{2}\mathbb{Z}$-gradation (1.1) are called *good* if one of the equivalent properties (1.10) or (1.11) holds. In this case the $\mathfrak{g}^{\natural}$-modules $\mathfrak{g}_{-1/2}$ and $\mathfrak{g}_{1/2}$ are isomorphic. Furthermore, we have the exact sequence of $\mathfrak{g}^{\natural}$-modules:

$$0 \to \mathfrak{g}^f \to \mathfrak{g}_- + \mathfrak{g}_0 + \mathfrak{g}_{1/2} \xrightarrow{\operatorname{ad} f} \mathfrak{g}_- \to 0.$$

It follows that $\operatorname{ad} f$ induces the following isomorphism of $\mathfrak{g}^{\natural}$-modules:

$$\mathfrak{g}^f \xrightarrow{\sim} \mathfrak{g}_0 \oplus \mathfrak{g}_{1/2}. \tag{1.12}$$

One can find a detailed study and a classification of good gradations of simple Lie algebras in [EK].

Remark 1.1. Property (1.11) for $j=0$ gives: $[\mathfrak{g}_0, f] = \mathfrak{g}_{-1}$. In particular, it follows that f lies in the dense open orbit in $\mathfrak{g}_{-1}$ of the algebraic group $G_{0\bar{0}}$ whose Lie algebra is the even part of $\mathfrak{g}_0$. Thus, up to conjugacy, f is determined by x. Hence, instead of a good pair (x,f) one may talk about a good semi-simple element x, defined by the properties that the eigenvalues of $\operatorname{ad} x$ lie in $\frac{1}{2}\mathbb{Z}$ and the centralizer of any element f from the open orbit of $G_{0\bar{0}}$ in $\mathfrak{g}_{-1}$ lies in $\mathfrak{g}_{\leqslant}$.

2. Fields of the vertex algebra $W_k(\mathfrak{g}, x, f)$

The dual Coxeter number $h^\vee$, defined as the half of the eigenvalue of the Casimir operator on $\mathfrak{g}$, is given by

$$h^\vee = (\rho|\theta) + \tfrac{1}{2}(\theta|\theta), \tag{2.1}$$

where ρ is the half of the difference of the sum of positive even roots and the sum of positive odd roots of $\mathfrak{g}$, and θ is the highest root (for any choice of a set of positive roots).

Provided that $k \neq -h^\vee$, define the energy–momentum (or Virasoro) field

$$L(z) = L^{\mathfrak{g}}(z) + \frac{d}{dz}x(z) + L^{\mathrm{ch}}(z) + L^{\mathrm{ne}}(z). \tag{2.2}$$

Here

$$L^{\mathfrak{g}} = \frac{1}{2(k+h^\vee)} \sum_{\alpha \in S} (-1)^{p(\alpha)} :u_\alpha u^\alpha:$$

is the Sugawara construction, where $\{u^\alpha\}_{\alpha \in S}$ is the dual basis, i.e., $(u_\alpha|u^\beta) = \delta_{\alpha\beta}$,

$$L^{\mathrm{ch}} = -\sum_{\alpha \in S_+} m_\alpha :\varphi^\alpha \partial\varphi_\alpha: + \sum_{\alpha \in S_+} (1-m_\alpha) :(\partial\varphi^\alpha)\varphi_\alpha:, L^{\mathrm{ne}} = \frac{1}{2} \sum_{\alpha \in S_{1/2}} :(\partial\Phi^\alpha)\Phi_\alpha:.$$

The central charge of $L(z)$ equals [KRW]:

$$c(\mathfrak{g}, x, f, k) = \frac{k \,\mathrm{sdim}\, \mathfrak{g}}{k + h^\vee} - 12k(x|x) - \sum_{\alpha \in S_+} (-1)^{p(\alpha)} (12m_\alpha^2 - 12m_\alpha + 2) \\ - 1/2 \,\mathrm{sdim}\, \mathfrak{g}_{1/2}. \tag{2.3}$$

(Here and further sdim stands for the super-dimension of a super vector space.)

With respect to $L(z)$ the fields φ_α (resp. φ^α) are primary of conformal weight $1 - m_\alpha$ (resp. m_α), the fields Φ_α are primary of conformal weight $\frac{1}{2}$, and the fields $a(z)$ for $a \in \mathfrak{g}_j$ have conformal weight $1 - j$ and are primary unless $j = 0$ and $(x|a) \neq 0$. Furthermore, it was shown in [KRW] that the field $d(z)$ is primary of conformal weight 1, hence $[d_\lambda L] = \lambda d$ and $d_0(L) = [d_\lambda L]|_{\lambda=0} = 0$. Thus, the homology class of L defines the energy–momentum field of $W_k(\mathfrak{g}, x, f)$, which we again denote by L.

In order to construct some other fields of the vertex algebra $W_k(\mathfrak{g}, x, f)$, for $v \in \mathfrak{g}$ denote by $c_{\alpha\beta}(v)$ the matrix of $\mathrm{ad}\, v$ in the basis $\{u_\alpha\}_{\alpha \in S}$, i.e., $[v, u_\beta] = \sum_{\alpha \in S} c_\beta^\alpha(v) u_\alpha$. Given $v \in \mathfrak{g}_j$, introduce the following field of conformal weight $1 - j$:

$$J^{(v)}(z) = v(z) + \sum_{\alpha, \beta \in S_+} (-1)^{p(\alpha)} c_\beta^\alpha(v) : \varphi_\alpha(z) \varphi^\beta(z) : . \tag{2.4}$$

The fields $J^{(v)}$ will be the main building blocks for the vertex algebra $W_k(\mathfrak{g}, x, f)$. They obey the following λ-brackets [KRW]:

$$[J^{(v)}{}_\lambda J^{(v')}] = J^{([v,v'])} + \lambda(k(v|v') + \tfrac{1}{2}(\kappa_\mathfrak{g}(v, v') - \kappa_{\mathfrak{g}_0}(v, v'))) \tag{2.5}$$

if $v \in \mathfrak{g}_i$, $v' \in \mathfrak{g}_j$ and $ij \geqslant 0$, where $\kappa_\mathfrak{g}$ (resp. $\kappa_{\mathfrak{g}_0}$) denotes the Killing form on $\mathfrak{g}$ (resp. $\mathfrak{g}_0$).

The following important formula is established via the λ-bracket calculus, using formulas (2.4) from [KRW]:

$$\begin{aligned} d_0(J^{(v)}) = & \sum_{\beta \in S_+} ([f, v]|u_\beta) \varphi^\beta + \sum_{\beta \in S_+} (-1)^{p(v)(p(\beta)+1)} : \varphi^\beta \Phi_{[v, u_\beta]} : \\ & - \sum_{\substack{\beta \in S_+: \\ [v, u_\beta] \in \mathfrak{g}_\leqslant}} (-1)^{p(\beta)(p(v)+1)} : \varphi^\beta J^{([v, u_\beta])} : \\ & + \sum_{\beta \in S_+} (k(v|u_\beta) + \mathrm{str}_{\mathfrak{g}_+}\, p_+ (\mathrm{ad}\, v)(\mathrm{ad}\, u_\beta)) \partial \varphi^\beta, \end{aligned} \tag{2.6}$$

where p_+ is the projection of $\mathfrak{g}$ on $\mathfrak{g}_+$.

Furthermore, introduce the following fields of conformal weight 1 and $\frac{3}{2}$ for $v \in \mathfrak{g}_0$ and $v \in \mathfrak{g}_{-1/2}$, respectively:

$$v^{\mathrm{ne}} = \frac{(-1)^{p(v)}}{2} \sum_{\alpha \in S_{1/2}} : \Phi^\alpha \Phi_{[u_\alpha, v]} : (v \in \mathfrak{g}_0) \tag{2.7}$$

$$v^{\mathrm{ne}} = -\frac{(-1)^{p(v)}}{3} \sum_{\alpha,\beta \in S_{1/2}} :\Phi^{\alpha}\Phi^{\beta}\Phi_{[u_\beta,[u_\alpha,v]]}: (v \in \mathfrak{g}_{-1/2}). \tag{2.8}$$

Theorem 2.1. (a) *For* $v \in \mathfrak{g}_0$ *let* $J^{\{v\}} = J^{(v)} + v^{\mathrm{ne}}$. *Then, provided that* $v \in \mathfrak{g}^{\natural}$, *we have* $d_0(J^{\{v\}}) = 0$, *hence the homology class of* $J^{\{v\}}$ *defines a field of the vertex algebra* $W_k(\mathfrak{g}, x, f)$ *of conformal weight* 1.

(b) $[L_\lambda J^{(v)}] = (\partial + (1-j)\lambda)J^{(v)} + \delta_{j0}\lambda^2\big(\frac{1}{2}\mathrm{str}_{\mathfrak{g}_+}(\mathrm{ad}\, v) - (k+h^\vee)(v|x)\big)$ *if* $v \in \mathfrak{g}_j$, *and the same formula holds for* $J^{\{v\}}$ *if* $v \in \mathfrak{g}_0$.

(c) $[J^{\{v\}}{}_\lambda J^{\{v'\}}] = J^{\{[v,v']\}} + \lambda(k(v|v') + \frac{1}{2}(\kappa_{\mathfrak{g}}(v,v') - \kappa_{\mathfrak{g}_0}(v,v') - \kappa_{1/2}(v,v')))$ *provided that* $v, v' \in \mathfrak{g}_0^f$, *where* $\kappa_{1/2}$ *is the supertrace form of* $\mathfrak{g}_0$ *on* $\mathfrak{g}_{1/2}$.

(d) *For* $v \in \mathfrak{g}_{-1/2}$ *let*

$$\begin{aligned} G^{\{v\}} = {} & J^{(v)} + v^{\mathrm{ne}} + \sum_{\beta \in S_{1/2}} :J^{([v,u_\beta])}\Phi^{\beta}: \\ & - \sum_{\beta \in S_{1/2}} (k(v|u_\beta) + \mathrm{str}_{\mathfrak{g}_+}(\mathrm{ad}\, v)(\mathrm{ad}\, u_\beta))\partial\Phi^{\beta}. \end{aligned}$$

Then provided that $v \in \mathfrak{g}_{-1/2}^f$, *we have* $d_0(G^{\{v\}}) = 0$, *hence the homology class of* $G^{\{v\}}$ *defines a field of the vertex algebra* $W_k(\mathfrak{g}, x, f)$ *of conformal weight* $\frac{3}{2}$. *This field is primary.*

(e) *Provided that* $a \in \mathfrak{g}^{\natural}$ *and* $v \in \mathfrak{g}_{-1/2}^f$, *we have*:

$$[J^{\{a\}}{}_\lambda G^{\{v\}}] = G^{\{[a,v]\}}.$$

Proof. (a), (b) and (c) were proved in [KRW], Theorem 2.4, by making use of the λ-bracket calculus. The proof of (d) and (e) is similar. In the calculation of the λ-bracket of L with $G^{\{v\}}$ $(v \in \mathfrak{g}_{-1/2}^f)$ we find: $[L_\lambda G^{\{v\}}] = (\partial + \frac{3}{2}\lambda)G^{\{v\}} + \lambda^2 \sum_\alpha c_\alpha \Phi^\alpha$, $c_\alpha \in \mathbb{C}$. Since $d_0(L) = 0$, $d_0(G^{\{v\}}) = 0$ and $d_0(\Phi^\alpha) = \varphi^\alpha$, we conclude that all $c_\alpha = 0$. $\square$

3. A digression to homology

In this section we collect some general facts about homology that will be used in the sequel.

Let V be a vector superspace with an odd endomorphism d such that $d^2 = 0$, and let $H(V, d) = \mathrm{Ker}\, d/\mathrm{Im}\, d$ denote the *homology* of the *complex* (V, d). The following well-known lemma is very useful.

Lemma 3.1 (Künneth lemma). *Let (V, d) be a complex, and suppose that $V = V_1 \otimes V_2$ and $d = d_1 \otimes 1 + 1 \otimes d_2$. Then the canonical map $H(V_1, d_1) \otimes H(V_2, d_2) \to H(V, d)$ is a vector space isomorphism.*

The next lemma is probably well known too.

Lemma 3.2. *Let $\mathfrak{g}$ be a Lie superalgebra and let d be an odd derivation of $\mathfrak{g}$ such that $d^2 = 0$. Extend d to an odd derivation of $U(\mathfrak{g})$. Then $H := H(\mathfrak{g}, d)$ is a Lie superalgebra too and the canonical homomorphism of $U(H)$ to $H(U(\mathfrak{g}), d)$ is an isomorphism of associative algebras. In particular, if $H = 0$, then $H(U(\mathfrak{g}), d) = \mathbb{C}1$.*

Proof. First, it follows from the Künneth lemma that

$$H(S^n(\mathfrak{g}), d) = S^n(H(\mathfrak{g})) \quad \text{for any } n \in \mathbb{Z}. \tag{3.1}$$

Indeed, by the Künneth lemma, (3.1) holds if S is replaced by T. But since the action of the symmetric group S_n on $\mathfrak{g}^{\otimes n}$ commutes with the action of d, (3.1) holds too.

Next, consider the exact sequence of vector spaces:

$$0 \to U_{n-1}(\mathfrak{g}) \to U_n(\mathfrak{g}) \to S^n(\mathfrak{g}) \to 0,$$

where $\{U_n(\mathfrak{g})\}_n$ is the increasing PBW filtration of $U(\mathfrak{g})$. This exact sequence induces a long exact sequence of homology:

$$H(S^n(\mathfrak{g}), d) \xrightarrow{\delta} H(U_{n-1}(\mathfrak{g}), d) \to H(U_n(\mathfrak{g}), d) \to H(S^n(\mathfrak{g}), d) \xrightarrow{\delta} H(U_{n-1}(\mathfrak{g}), d),$$

where δ is the boundary map. Recall the construction of δ. Take $w \in H(S^n(\mathfrak{g}), d)$. By (3.1), we can choose a representative $\tilde{w}$ of w, which is a polynomial in the closed elements of $\mathfrak{g}$. Take the preimage $\tilde{\tilde{w}}$ of $\tilde{w}$ in $U_n(\mathfrak{g})$ via the supersymmetrization map. Then the class of $d\tilde{\tilde{w}}$ lies in $H(U_{n-1}(\mathfrak{g}), d)$ and we let $\delta(w) = d\tilde{\tilde{w}}$. But $d\tilde{\tilde{w}} = 0$, hence $\delta = 0$ and we have the exact sequence

$$0 \to H(U_{n-1}(\mathfrak{g}), d) \to H(U_n(\mathfrak{g}), d) \to H(S^n(\mathfrak{g}), d) \to 0.$$

By induction on n, this proves the lemma using (3.1). $\square$

Lemma 3.3. *Let $0 \to \mathfrak{c} \to \tilde{\mathfrak{g}} \to \mathfrak{g} \to 0$ be an extension of Lie superalgebras and let d be an odd derivation of $\tilde{\mathfrak{g}}$ such that $d(\mathfrak{c}) = 0$, so that d induces a derivation d of $\mathfrak{g}$, and suppose that $d(\tilde{\mathfrak{g}})$ is contained in an ideal J of $\tilde{\mathfrak{g}}$ such that $J \cap \mathfrak{c} = 0$. Let $\lambda : \mathfrak{c} \to \mathbb{C}$ be a Lie superalgebra homomorphism and let $\mathfrak{c}_\lambda \subset \tilde{\mathfrak{g}} + \mathbb{C} \cdot 1$ denote the subspace $\{c - \lambda(c) \mid c \in \mathfrak{c}\}$. Then*

$$H(U(\tilde{\mathfrak{g}}), d)/(\mathfrak{c}_\lambda) = U(H(\tilde{\mathfrak{g}}, d))/(\mathfrak{c}_\lambda).$$

Proof. Include J in a complementary subspace $\mathfrak{g}$ to $\mathfrak{c}$ in $\tilde{\mathfrak{g}}$, choose a basis of J, extend it to a basis of $\mathfrak{g}$ and denote by $U(\mathfrak{g})$ the span of the PBW monomials in this basis. Then $U(\mathfrak{g})$ is a d-invariant subspace of $U(\mathfrak{g})$ and we have a vector space decomposition: $U(\tilde{\mathfrak{g}}) = U(\mathfrak{c}) \otimes U(\mathfrak{g})$. Hence, by the Künneth lemma we have:

$$H(U(\tilde{\mathfrak{g}}), d) = U(\mathfrak{c}) \otimes H(U(\mathfrak{g}), d) \text{ (tensor product of vector spaces)},$$

and the lemma follows. $\square$

Lemma 3.4. (a) *Let R be a Lie conformal superalgebra with an odd derivation d (i.e., $d[a_\lambda b] = [da_\lambda b] + (-1)^{p(a)}[a_\lambda db]$) such that $d^2 = 0$. Extend d to a derivation of the universal enveloping vertex algebra $V(R)$. Then $H := H(R, d)$ is a Lie conformal superalgebra too, and the canonical homomorphism of its universal enveloping vertex algebra $V(H)$ to $H(V(R), d)$ is an isomorphism of vertex algebras. In particular, if $H = 0$, then $H(V(R), d) = \mathbb{C}|0\rangle$.*

(b) *Lemma 3.3 holds if we replace Lie superalgebras by Lie conformal superalgebras and universal enveloping algebras by universal enveloping vertex algebras.*

Proof. Since $V(R)$ is identified with $U(R_-)$, where R_- is the space R with the Lie superalgebra bracket $[a, b] = \int_{-\partial}^{0} [a_\lambda b] d\lambda$ [BK,GMS], Lemma 3.4 follows from Lemmas 3.2 and 3.3. $\square$

4. The structure of the vertex algebra $W_k(\mathfrak{g}, x, f)$

Denote by $\mathscr{C}^+$ the vertex subalgebra of the vertex algebra $\mathscr{C}(\mathfrak{g}, x, f, k)$ generated by the fields φ_α and $d_0(\varphi_\alpha)$ for all $\alpha \in S_+$. From [KRW], formula (2.4), we have:

$$d_0(\varphi_\alpha) = \begin{cases} J^{(u_\alpha)} + (-1)^{p(\alpha)} \Phi_\alpha & \text{if } \alpha \in S_{1/2}, \\ J^{(u_\alpha)} + (f|u_\alpha) & \text{if } \alpha \in S_+ \backslash S_{1/2}. \end{cases}$$

Using this and [KRW], Lemma 2.1(a) we get

$$[d_0(\varphi_\alpha)_\lambda \varphi_\beta] = (-1)^{p(\alpha)} \sum_\gamma c_\beta^\alpha(u_\alpha)\varphi_\gamma.$$

It follows that

$$R = \sum_{\alpha \in S_+} \mathbb{C}[\partial]\varphi_\alpha + \sum_{\alpha \in S_+} \mathbb{C}[\partial] d_0(\varphi_\alpha)$$

is closed under the λ-bracket and is d_0-invariant. Hence we have a Lie conformal algebra complex (R, d_0), and the homology of this complex is obviously zero. It is also clear that $\mathscr{C}^+$ is the universal enveloping vertex algebra of R. Applying

Lemma 3.2, we obtain:

$$H(\mathscr{C}^+, d_0) = \mathbb{C}|0\rangle. \tag{4.1}$$

Next, denote by $\mathscr{C}^-$ the vertex subalgebra of the vertex algebra $\mathscr{C}(\mathfrak{g}, x, f, k)$ generated by the fields $J^{(u)}$ for all $u \in \mathfrak{g}_{\leqslant}$, the fields φ^α for all $\alpha \in S_+$ and the fields Φ_α for all $\alpha \in S_{1/2}$. Then obviously we have:

$$\mathscr{C}(\mathfrak{g}, x, f, k) = \mathscr{C}^+ \otimes \mathscr{C}^- \text{ (as vector spaces)}. \tag{4.2}$$

Recall that ([KRW], formula (2.4)):

$$d_0(\varphi^\alpha) = -\frac{1}{2} \sum_{\beta,\gamma \in S_+} (-1)^{p(\alpha)p(\beta)} c^\alpha_{\beta\gamma} \varphi^\beta \varphi^\gamma, \tag{4.3}$$

$$d_0(\Phi_\beta) = \sum_{\alpha \in S_{1/2}} \langle u_\alpha, u_\beta \rangle_{\mathrm{ne}} \varphi^\alpha, \quad d_0(\Phi^\beta) = \varphi^\beta. \tag{4.4}$$

It follows from formulas (2.6), (4.3) and(4.4) that $\mathscr{C}^-$ is d_0-invariant. Hence by the Künneth lemma, (4.1) and (4.2) imply

$$W_k(\mathfrak{g}, x, f, k) = H(\mathscr{C}^-, d_0). \tag{4.5}$$

In order to compute the homology of the complex $(\mathscr{C}^-, d_0)$, consider the following Lie conformal subalgebra of the vertex algebra $\mathscr{C}^-$:

$$\tilde{R} = \sum_{v \in \mathfrak{g}_{\leqslant}} \mathbb{C}[\partial] J^{(v)} + \sum_{\alpha \in S_+} \mathbb{C}[\partial]\varphi^\alpha + \sum_{\alpha \in S_{1/2}} \mathbb{C}[\partial]\Phi^\alpha + \mathbb{C}|0\rangle,$$

and define an odd $\mathbb{C}[\partial]$-module endomorphism d_1 of $\tilde{R}$ by (cf. (2.6)):

$$d_1(J^{(v)}) = \sum_{\beta \in S_+} ([f, v]|u_\beta)\varphi^\beta, \quad d_1(\varphi^\alpha) = 0, \quad d_1(\Phi^\alpha) = \varphi^\alpha, \quad d_1(|0\rangle) = 0. \tag{4.6}$$

It is straightforward to check that d_1 is an (odd) derivation of the Lie conformal superalgebra $\tilde{R}$, and it is obvious that $d_1^2 = 0$.

Next, let $d_2 = d_0 - d_1$. This is, of course, an odd derivation of the vertex algebra $\mathscr{C}^-$, since both d_0 and d_1 are.

Lemma 4.1. $d_2^2 = 0$ *and* $d_1 d_2 + d_2 d_1 = 0$.

Proof. We have: $0 = d_0^2 = d_1^2 + (d_1 d_2 + d_2 d_1) + d_2^2$. Hence it remains to show that $d_1 d_2 + d_2 d_1 = 0$. We have the following formulas for d_2:

$$d_2(J^{(v)}) = \sum_{\alpha \in S_+} (-1)^{p(v)(p(\alpha)+1)} \varphi^\alpha \Phi_{[v,u_\alpha]} - \sum_{\substack{\alpha \in S_+: \\ [v,u_\alpha] \in \mathfrak{g}_\leqslant}} (-1)^{p(\alpha)(p(v)+1)} : \varphi^\alpha J^{([v,u_\alpha])} :$$

$$+ \sum_{\beta \in S_+} (k(v|u_\beta) + \mathrm{str}_{\mathfrak{g}_+} p_+(\mathrm{ad}\, v)(\mathrm{ad}\, u_\beta)) \partial \varphi^\beta,$$

$$d_2(\varphi^\gamma) = -\frac{1}{2} \sum_{\alpha,\beta \in S_+} (-1)^{p(\alpha)p(\gamma)} c^\gamma_{\alpha\beta} \varphi^\alpha \varphi^\beta, \quad d_2(\Phi_u) = 0. \tag{4.7}$$

It follows from (4.6) and (4.7) that $d_1 d_2 + d_2 d_1$ annihilates the φ^α and the Φ_α. Hence it remains to check that

$$(d_1 d_2 + d_2 d_1) J^{(v)} = 0 \quad \text{for } v \in \mathfrak{g}_\leqslant.$$

Using (4.6) and (4.7), we obtain:

$$\begin{aligned} d_2 d_1 (J^{(v)}) &= -\frac{1}{2} \sum_{\alpha,\beta,\gamma \in S_+} (-1)^{p(\alpha)p(\gamma)} c^\gamma_{\alpha\beta} (f\,|[v, u_\gamma]) \varphi^\alpha \varphi^\beta \\ &= -\frac{1}{2} \sum_{\alpha,\beta \in S_+} (-1)^{p(\alpha)(p(\beta)+1)} (f\,|[v, [u_\alpha, u_\beta]]) \varphi^\alpha \varphi^\beta, \end{aligned}$$

since $p(\gamma) = p(\alpha) + p(\beta)$ if $c^\gamma_{\alpha\beta} \neq 0$. Using the Jacobi identity, we get: $d_2 d_1 (J^{(v)}) = \frac{1}{2} A + \frac{1}{2} B$, where

$$A = -\sum_{\alpha,\beta \in S_+} (-1)^{p(\alpha)(p(\beta)+1)} (f\,|[[v, u_\alpha], u_\beta]) \varphi^\alpha \varphi^\beta,$$

$$B = \sum_{\alpha,\beta \in S_+} (-1)^{p(\alpha)} (f\,|[[v, u_\beta], u_\alpha]) \varphi^\alpha \varphi^\beta.$$

Exchanging α and β in A and using that $\varphi^\beta \varphi^\alpha = (-1)^{(p(\alpha)+1)(p(\beta)+1)} \varphi^\alpha \varphi^\beta$, we see that $A = B$, hence

$$d_2 d_1 (J^{(v)}) = B.$$

Next, using again (4.6) and (4.7), we obtain: $d_1 d_2 (J^{(v)}) = C + D$, where

$$C = \sum_{\beta \in S_+} (-1)^{p(v)(p(\beta)+1)} d_1 (: \varphi^\beta \Phi_{[v,u_\beta]} :),$$

$$D = -\sum_{\substack{\beta \in S_+: \\ [v,u_\beta] \in \mathfrak{g}_\leqslant}} (-1)^{p(\beta)(p(v)+1)} d_1 (: \varphi^\beta J^{([v,u_\beta])} :).$$

In the calculation of C we use the formula $\Phi_u = \sum_{\alpha \in S_{1/2}} (f\,|[u_\alpha, u])\Phi^\alpha$ to obtain:

$$C = \sum_{\substack{\beta \in S_+: \\ [v,u_\beta] \in \mathfrak{g}_{1/2}}} \sum_{\alpha \in S_{1/2}} (-1)^{(p(v)+1)(p(\beta)+1)} (f\,|[u_\alpha, [v, u_\beta]]) \varphi^\beta \varphi^\alpha$$
$$= \sum_{\substack{\beta \in S_+: \\ [v,u_\beta] \in \mathfrak{g}_\leqslant}} \sum_{\alpha \in S_{1/2}} (-1)^{p(\alpha)} (f\,|[[v, u_\beta], u_\alpha]) \varphi^\alpha \varphi^\beta.$$

Finally, we have:

$$D = \sum_{\substack{\beta \in S_+: \\ [v,u_\beta] \in \mathfrak{g}_\leqslant}} (-1)^{p(v)p(\beta)} \sum_{\alpha \in S_+} (f\,|[[v, u_\beta], u_\alpha]) \varphi^\beta \varphi^\alpha$$
$$= -\sum_{\substack{\beta \in S_+: \\ [v,u_\beta] \in \mathfrak{g}_\leqslant}} \sum_{\alpha \in S_+} (-1)^{p(\alpha)} (f\,|[[v, u_\beta], u_\alpha]) \varphi^\alpha \varphi^\beta.$$

Hence $(d_1 d_2 + d_2 d_1)J^{(v)} = B + C + D = 0$ since $(f\,|[[v, u_\beta], u_\alpha]) = 0$ if $[[v, u_\beta], u_\alpha] \in \mathfrak{g}_j$ with $j \neq 1$. This completes the proof. $\square$

From now on assume that the pair (x, f) is good (recall that this is the case if the pair can be included in an $s\ell_2$-triple). Then, by (1.11), for each $r \geqslant 1$, $r \in \frac{1}{2}\mathbb{Z}$ we can choose elements $u^\alpha \in \mathfrak{g}_{1-r} (\alpha \in S_r)$ such that

$$(f\,|[u^\alpha, u_\beta]) = \delta_{\alpha\beta} \quad \text{for all } \alpha, \beta \in S_r.$$

Note that

$$\mathfrak{g}_\leqslant = \sum_{\alpha \in S_+ \backslash S_{1/2}} \mathbb{C} u^\alpha \oplus \mathfrak{g}^f_\leqslant. \tag{4.8}$$

Then we have from (4.6):

$$d_1(J^{(u^\alpha)}) = \varphi^\alpha, \quad \alpha \in S_+ \backslash S_{1/2}. \tag{4.9}$$

It follows from (4.6), (4.8) and (4.9) that

$$H(\tilde{R}, d_1) = \mathbb{C}[\partial]\mathfrak{g}^f_\leqslant + \mathbb{C}|0\rangle, \tag{4.10}$$

where on the right we have a central extension of the Lie conformal superalgebra $\mathbb{C}[\partial]\mathfrak{g}^f_\leqslant$ defined by $[a_\lambda b] = [a, b] + \psi_k(a, b)\lambda$, $a, b \in \mathfrak{g}^f_\leqslant$, and the 2-cocycle ψ_k is given by (cf. (2.5)):

$$\psi_k(a, b) = k(a|b) + \tfrac{1}{2}(\kappa_{\mathfrak{g}}(a, b) - \kappa_{\mathfrak{g}_0}(a, b)).$$

Let $\widehat{\mathfrak{g}^f_\leqslant}$ denote the central extension of the Lie superalgebra $\mathfrak{g}^f_\leqslant[t, t^{-1}]$ corresponding to the cocycle $\alpha_k(at^m, bt^n) = m\delta_{m,-n}\psi_k(a, b)$ and let $V_{\alpha_k}(\mathfrak{g}^f)$ denote the corresponding universal affine vertex algebra. By Lemma 3.4(b), (4.10) gives

$$H(\mathscr{C}^-, d_1) \simeq V_{\alpha_k}(\mathfrak{g}^f_\leqslant).$$

Equivalently, this can be formulated as the following lemma:

Lemma 4.2. *The vertex algebra $H(\mathscr{C}^-, d_1)$ is a vertex subalgebra of the vertex algebra $\mathscr{C}^-$, strongly generated by the fields $J^{(v)}$, where $v \in \mathfrak{g}^f_\leqslant$.*

Now we can state and prove the main theorem on the structure of the vertex algebra $W_k(\mathfrak{g}, x, f)$.

Theorem 4.1. *Let $\mathfrak{g}$ be a simple finite-dimensional Lie superalgebra with an invariant bilinear form $(.|.)$, and let x, f be a pair of even elements of $\mathfrak{g}$ such that $\mathrm{ad}\, x$ is diagonalizable with eigenvalues in $\frac{1}{2}\mathbb{Z}$ and $[x, f] = -f$. Suppose that all eigenvalues of $\mathrm{ad}\, x$ on $\mathfrak{g}^f$ are non-positive: $\mathfrak{g}^f = \oplus_{j \leqslant 0} \mathfrak{g}^f_j$. Then*

(a) *For each $a \in \mathfrak{g}^f_{-j}$ $(j \geqslant 0)$ there exists a d_0-closed field $J^{\{a\}}$ in $\mathscr{C}^-$ of conformal weight $1 + j$ (with respect to L) such that $J^{\{a\}} - J^{(a)}$ is a linear combination of normal ordered products of the fields $J^{(b)}$, where $b \in \mathfrak{g}_{-s}$, $0 \leqslant s < j$, the fields Φ_α, where $\alpha \in S_{1/2}$, and the derivatives of these fields.*

(b) *The homology classes of the fields $J^{\{a_i\}}$, where $a_1, a_2, \ldots$ is a basis of $\mathfrak{g}^f$ compatible with its $\frac{1}{2}\mathbb{Z}$-gradation, strongly generate the vertex algebra $W_k(\mathfrak{g}, x, f)$ and obey the PBW theorem (see Remark 4.2 below).*

(c) *$H_0(\mathscr{C}(\mathfrak{g}, x, f, k), d_0) = W_k(\mathfrak{g}, x, f)$ and $H_j(\mathscr{C}(\mathfrak{g}, x, f, k), d_0) = 0$ if $j \neq 0$.*

Proof. Consider the $\left(\frac{1}{2}\mathbb{Z}\right)^2$-gradation $\mathscr{C}^- = \oplus_{m,n \in \frac{1}{2}\mathbb{Z}} \mathscr{C}^-_{m,n}$ defined by the following relations:

$$\deg|0\rangle = (0, 0), \quad \deg J^{(v)} = (-j, j) \text{ if } v \in \mathfrak{g}_{-j} \quad (j \in \tfrac{1}{2}\mathbb{Z}_+),$$

$$\deg \Phi_\alpha = \left(\tfrac{1}{2}, -\tfrac{1}{2}\right)(\alpha \in S_{1/2}), \quad \deg \varphi^\alpha = (-m_\alpha, m_\alpha - 1) \quad \text{if } u_\alpha \in \mathfrak{g}_{m_\alpha}.$$

It is clear from (4.6) and (4.7) that $d_1\mathscr{C}_{m,n} \subset \mathscr{C}_{m-1,n}$, $d_2\mathscr{C}_{m,n} \subset \mathscr{C}_{m,n-1}$. Thus, by Lemma 4.1, $\mathscr{C}^-$ is a bicomplex. By Lemma 4.2, $H(\mathscr{C}^-, d_1) \subset \oplus_{m \in \frac{1}{2}\mathbb{Z}_+} \mathscr{C}^-_{-m,m}$. Hence d_2 induces a zero differential on $H(\mathscr{C}^-, d_1)$ and all higher differentials of the spectral sequence are zero. Note also that both d_1 and d_2 preserve the conformal weights, i.e., the eigenvalues of L_0. Since the eigenspaces of L_0 on $\mathscr{C}^-$ are finite-dimensional, the bicomplex $\mathscr{C}^-$ is locally finite.

Hence, given $a \in \mathfrak{g}^f_{-j}$ $(j \in \mathbb{Z}_+)$ the closed field $J^{(a)}$ with respect to d_1 can be extended to a closed field $J^{\{a\}}$ with respect to d_0, of the form:

$$J^{\{a\}} = J^{(a)} + \sum_r A_{-j+r,\, j-r} \text{ (finite sum, } r \in \mathbb{Z}, r \geqslant 1). \tag{4.11}$$

By the total degree considerations, the fields $A_{-j+r,\,j-r} \in \mathscr{C}^-_{-j+r,\,j-r}$ do not involve the fields φ^α, and hence they are linear combinations of normally ordered products of the fields $J^{(b)}$, Φ_α, and their derivatives. This proves (a). The statement (b) follows from (a) by applying the Künneth lemma and (4.1). The statement (c) follows from (b). □

Remark 4.1. The equation $(d_1 + d_2)J^{\{a\}} = 0$ is equivalent to the following system of equations on the $A_{-j+r,\,j-r}$ in (4.11):

$$d_1(A_{-j+1,\,j-1}) = -d_2(J^{(a)}), \quad d_1(A_{-j+r,\,j-r}) = d_2(A_{-j+r-1,\,j-r+1}) \quad \text{for } r \geqslant 2. \tag{4.12}$$

Of course the fields of conformal weight 1 and $\frac{3}{2}$ of the vertex algebra $W_k(\mathfrak{g}, x, f)$ constructed in Theorem 2.1(a) and (d), respectively, are of the form given by Theorem 4.1(a). It is easy to see that they are the only solutions of the system (4.12) for a given $a \in \mathfrak{g}^f_0$ (resp. $\mathfrak{g}^f_{-1/2}$). However, for $j \leqslant -1$ the solution is not unique, and it is more difficult to write down a solution explicitly.

Remark 4.2. Write each field $J^{\{a_i\}}$ in the form $J^{\{a_i\}}(z) = \sum_{n \in \mathbb{Z} - \Delta_i} J^{\{i\}}_n z^{-n-\Delta_i}$, where Δ_i is the conformal weight. Then the property that the $J^{\{a_i\}}$ strongly generate the vertex algebra $W_k(\mathfrak{g}, x, f)$ means that the monomials,

$$\begin{aligned} &(J^{\{i_1\}}_{-m_1})^{b_1} (J^{\{i_2\}}_{-m_2})^{b_2} \dots (J^{\{i_s\}}_{-m_s})^{b_s} |0\rangle \text{ where } b_i \in \mathbb{Z}_+, b_i \leqslant 1 \\ &\text{if } a_i \text{ is odd, and } m_i > 1 - \Delta_i, \end{aligned} \tag{4.13}$$

span this vertex algebra. The property that they obey the PBW theorem means that the monomials (4.13), where the sequence of pairs (i_1, m_1), (i_2, m_2), … is decreasing in the lexicographical order, form a basis of $W_k(\mathfrak{g}, x, f)$.

5. The structure of vertex algebras $W_k(\mathfrak{g}, e_{-\theta})$, associated to minimal gradations

A $\frac{1}{2}\mathbb{Z}$-gradation of the Lie superalgebra $\mathfrak{g}$ is called *minimal* if it has the form

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-1/2} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{1/2} \oplus \mathfrak{g}_1 \tag{5.1}$$

and satisfies the following additional properties:

(a) $\mathfrak{g}_1 = \mathbb{C}e$ and $\mathfrak{g}_{-1} = \mathbb{C}f$, where e and f are even non-zero elements,
(b) (5.1) is the eigenspace decomposition with respect to $\operatorname{ad} x$, where $x = [e, f]$.

Since $\{e, x, f\}$ is an $s\ell_2$-triple, the minimal gradation is determined, up to conjugation, by the nilpotent element f. Note that $\mathfrak{g}^\natural$ is the centralizer of this triple, and we have:

$$\mathfrak{g}^\natural = \{a \in \mathfrak{g}_0 \mid (x|a) = 0\}, \quad \mathfrak{g}^f = \mathfrak{g}_{-1} + \mathfrak{g}_{-1/2} + \mathfrak{g}^\natural. \tag{5.2}$$

Choose a Cartan subalgebra $\mathfrak{h}$ of the even part of $\mathfrak{g}_0$. Then $\mathfrak{h}$ is a Cartan subalgebra of the even part of $\mathfrak{g}$ and $x \in \mathfrak{h}$, so that $\mathfrak{h}^\natural := \{h \in \mathfrak{h} \mid (x|h) = 0\}$ is the Cartan subalgebra of the even part of $\mathfrak{g}^\natural$. In particular, we have

$$\mathfrak{h} = \mathfrak{h}^\natural \oplus \mathbb{C}x. \tag{5.3}$$

Let $\Delta \subset \mathfrak{h}^*$ be the set of roots of $\mathfrak{h}$ in $\mathfrak{g}$. Choose a subset of positive roots $\Delta_+ \subset \Delta$ such that $\alpha(x) \geqslant 0$ if $\alpha \in \Delta_+$, and let θ be the highest root. Hence $e = e_\theta$ and $f = e_{-\theta}$ are root vectors attached to θ and $-\theta$.

It follows from the classification of simple finite-dimensional Lie superalgebras [K1] that such a Lie superalgebra $\mathfrak{g}$ carries a non-degenerate even supersymmetric invariant bilinear form and is not a Lie algebra iff $\mathfrak{g}$ is isomorphic to $s\ell(m|n)/\delta_{m,n}\mathbb{C}I$ $(m, n \geqslant 1, (m, n) \neq (1, 1))$, $osp(m|n) = spo(n|m)$ $(m \geqslant 1, n \geqslant 2$ even), $D(2, 1; a)$, $F(4)$, $G(3)$ or $H(n)$ $(n \geqslant 6$ even). It is easy to see that in the case of $H(n)$ the gradation (5.1) corresponding to $e_{-\theta}$ has $\dim \mathfrak{g}_{\pm 1} \geqslant 2$, hence does not satisfy (a). In all the remaining cases the even part of $\mathfrak{g}$ is reductive. Thus, θ is the highest root of one of the simple components of the even part of $\mathfrak{g}$ (which is $\mathfrak{g}$ if $\mathfrak{g}$ is a simple Lie algebra), and therefore the adjoint orbit of e (which coincides with that of f) in this simple component is the unique non-zero nilpotent orbit of minimal dimension in this simple component.

Conversely, if θ is an even highest root of $\mathfrak{g}$ for some ordering of the set of roots, choosing a root vector $e \in \mathfrak{g}_0$ and embedding e in an $s\ell_2$-triple $\{e, x, f\}$, we obtain a minimal $\frac{1}{2}\mathbb{Z}$-gradation given by the eigenspaces of $\operatorname{ad} x$. Thus, up to conjugation, minimal $\frac{1}{2}\mathbb{Z}$-gradations of $\mathfrak{g}$ are classified by simple components of the even part of $\mathfrak{g}$ whose highest root can be made a highest root of $\mathfrak{g}$ for some ordering of Δ. It is easy to see, using the description of Δ in [K1], that this is always possible except when $\mathfrak{g} = osp(3|n)$ and the simple component of its even part is so_3.

We normalize the invariant bilinear form on $\mathfrak{g}$ by the condition $(\theta|\theta) = 2$. This determines uniquely the Casimir operator of $\mathfrak{g}$ and hence its eigenvalue $2h^\vee$ on $\mathfrak{g}$. The number $h^\vee$ is called the dual Coxeter number of the gradation (1.1). We shall identify $\mathfrak{h}$ with $\mathfrak{h}^*$ using this form. Then we have: $x = \theta/2$.

Since $[e, f] = \frac{1}{2}\theta$, we obtain $(e|f) = \frac{1}{2}$, and since $[a, b] \in \mathbb{C}e$ for $a, b \in \mathfrak{g}_{1/2}$, we get

$$[a, b] = 2\langle a, b\rangle_{\text{ne}} e \quad (a, b \in \mathfrak{g}_{1/2}). \tag{5.4}$$

Since in general [KW3]

$$\operatorname{str}_{\mathfrak{g}}(\operatorname{ad} a)(\operatorname{ad} b) = 2h^\vee (a|b), \quad a, b \in \mathfrak{g}, \tag{5.5}$$

and $(x|x) = 1/2$, we conclude by letting $a = b = x$ in (5.5) that

$$\mathrm{sdim}\, \mathfrak{g}_{1/2} = 2h^\vee - 4. \tag{5.6}$$

(Note that this equality means that $\mathrm{sdim}\, [\mathfrak{g}, e] = 2h^\vee - 2$.)

The above discussion gives a complete list of minimal gradations of $\mathfrak{g}$, along with $\mathfrak{g}^\natural$, $h^\vee$ and the description of the $\mathfrak{g}^\natural$-module $\mathfrak{g}_{1/2}$ $(\simeq \mathfrak{g}^*_{-1/2})$, which is given in Tables 1–3 (cf. [FL,KRW]):

Now we turn to a more detailed description of the vertex algebras $W_k(\mathfrak{g}, e_{-\theta})$ corresponding to minimal gradations of $\mathfrak{g}$. First, note that the central charge of the Virasoro field $L(z)$ defined by (2.2) is given by the following formula (we use (5.6) here):

$$c(\mathfrak{g}, e_{-\theta}, k) = \frac{k\, \mathrm{sdim}\, \mathfrak{g}}{k + h^\vee} - 6k + h^\vee - 4. \tag{5.7}$$

Note that

$$\mathrm{sdim}\, \mathfrak{g} = \mathrm{sdim}\, \mathfrak{g}^\natural + 2\, \mathrm{sdim}\, \mathfrak{g}_{1/2} + 3 = \mathrm{sdim}\, \mathfrak{g}^\natural + 4h^\vee - 5. \tag{5.8}$$

Let Ω_0 be the Casimir operator of $\mathfrak{g}_0$ for the bilinear form $(.\,|\,.)$ restricted to $\mathfrak{g}_0$. Since the $\mathfrak{g}_0$-module $\mathfrak{g}_{-1/2}$ (and $\mathfrak{g}_{1/2}$) is either irreducible or is a direct sum of two contragradient modules, Ω_0 has only one eigenvalue on $\mathfrak{g}_{-1/2}$ (equal that on $\mathfrak{g}_{1/2}$), which we denote by $h_{1/2}$.

Table 1
$\mathfrak{g}$ is a simple Lie algebra

$\mathfrak{g}$	$\mathfrak{g}^\natural$	$\mathfrak{g}_{1/2}$	$h^\vee$	$\mathfrak{g}$	$\mathfrak{g}^\natural$	$\mathfrak{g}_{1/2}$	$h^\vee$
$s\ell_n$ $(n\geqslant 3)$	$g\ell_{n-2}$	$\mathbb{C}^{n-2} \oplus \mathbb{C}^{n-2*}$	n	F_4	sp_6	$\Lambda_0^3\mathbb{C}^6$	9
so_n $(n\geqslant 5)$	$s\ell_2 \oplus so_{n-4}$	$\mathbb{C}^2 \otimes \mathbb{C}^{n-4}$	$n-2$	E_6	$s\ell_6$	$\Lambda^3\mathbb{C}^6$	12
sp_n $(n\geqslant 2)$	sp_{n-2}	$\mathbb{C}^{n-2}$	$\frac{n}{2}+1$	E_7	so_{12}	$spin_{12}$	18
G_2	$s\ell_2$	$S^3\mathbb{C}^2$	4	E_8	E_7	56-dim	30

Table 2
$\mathfrak{g}$ is not a Lie algebra, but $\mathfrak{g}^\natural$ is and $\mathfrak{g}_{1/2}$ is purely odd $(m\geqslant 1)$

$\mathfrak{g}$	$\mathfrak{g}^\natural$	$\mathfrak{g}_{1/2}$	$h^\vee$	$\mathfrak{g}$	$\mathfrak{g}^\natural$	$\mathfrak{g}_{1/2}$	$h^\vee$
$s\ell(2\|m)$ $(m\neq 2)$	$g\ell_m$	$\mathbb{C}^m \oplus \mathbb{C}^{m*}$	$2-m$	$D(2,1;a)$	$s\ell_2 \oplus s\ell_2$	$\mathbb{C}^2 \otimes \mathbb{C}^2$	0
$s\ell(2\|2)/\mathbb{C}I$	$s\ell_2$	$\mathbb{C}^2 \oplus \mathbb{C}^2$	0	$F(4)$	so_7	$spin_7$	-2
$spo(2\|m)$	so_m	$\mathbb{C}^m$	$2-\frac{m}{2}$	$G(3)$	G_2	7-dim	$-\frac{3}{2}$
$osp(4\|m)$	$s\ell_2 \oplus sp_m$	$\mathbb{C}^2 \otimes \mathbb{C}^m$	$2-m$				

Table 3
$\mathfrak{g}$ and $\mathfrak{g}^\natural$ are not Lie algebras ($m, n \geqslant 1$)

$\mathfrak{g}$	$\mathfrak{g}^\natural$	$\mathfrak{g}_{1/2}$	$h^\vee$
$s\ell(m\vert n)$ ($m \neq n, m > 2$)	$g\ell(m-2\vert n)$	$\mathbb{C}^{m-2\vert n} \oplus \mathbb{C}^{m-2\vert n*}$	$m-n$
$s\ell(m\vert m)/\mathbb{C}I$ ($m > 2$)	$s\ell(m-2\vert m)$	$\mathbb{C}^{m-2\vert m} \oplus \mathbb{C}^{m-2\vert m*}$	0
$spo(n\vert m)$ ($n \geqslant 4$)	$spo(n-2\vert m)$	$\mathbb{C}^{n-2\vert m}$	$\frac{n-m}{2}+1$
$osp(m\vert n)$ ($m \geqslant 5$)	$osp(m-4\vert n) \oplus s\ell_2$	$\mathbb{C}^{m-4\vert n} \otimes \mathbb{C}^2$	$m-n-2$
$F(4)$	$D(2,1;2)$	$\overset{1}{\bigcirc} \longleftarrow \otimes \longrightarrow \bigcirc$ (6|4)-dim	3
$G(3)$	$osp(3\vert 2)$	$\overset{-3}{\otimes} \Longrightarrow \overset{1}{\bigcirc}$ (4|4)-dim	2

Lemma 5.1. $h_{1/2} = h^\vee - 1$.

Proof. Let $S_j \subset \Delta$ denote the set of roots of $\mathfrak{h}$ in $\mathfrak{g}_j$ and let $\rho_0 = \frac{1}{2}\sum_{\alpha \in \Delta_+ \cap S_0} (-1)^{p(\alpha)}\alpha$. Then we have:

$$2\rho = 2\rho_0 + \sum_{\alpha \in S_{1/2}} (-1)^{p(\alpha)}\alpha + \theta. \tag{5.9}$$

It follows from (5.4) and the non-degeneracy of the form $\langle \,.\,, \,.\, \rangle_{\mathrm{ne}}$, that

$$\alpha \in S_{1/2} \quad \text{iff} \;\; \theta - \alpha \in S_{1/2}. \tag{5.10}$$

Let $\mu \in S_{1/2}$. Then, by (5.9), (5.10) and (5.6) we have:

$$\begin{aligned} 2(\mu|\rho) &= 2(\mu|\rho_0) + \frac{1}{2} \sum_{\alpha \in S_{1/2}} (-1)^{p(\alpha)} \left((\mu|\alpha) + (\theta - \mu|\alpha)\right) + (\mu|\theta) \\ &= 2(\mu|\rho_0) + \frac{1}{2} \operatorname{sdim} \mathfrak{g}_{1/2} + 1 = 2(\mu|\rho_0) + h^\vee - 1. \end{aligned}$$

Thus, we obtain

$$2(\mu|\rho - \rho_0) = h^\vee - 1. \tag{5.11}$$

Let $\Omega = \sum_{i=1}^r h^i h_i + 2\rho + 2\sum_{\alpha \in \Delta_+} e_{-\alpha} e_\alpha$ (resp. $\Omega_0 = \sum_{i=1}^r h^i h_i + 2\rho_0 + 2\sum_{\alpha \in \Delta_+ \cap S_0} e_{-\alpha} e_\alpha$) be the Casimir operator of $\mathfrak{g}$ (resp. of $\mathfrak{g}_0$), where $(e_\alpha | e_{-\alpha}) = 1$, $(h^i | h_j) = \delta_{ij}$. We have:

$$\Omega = 2(\rho - \rho_0) + \Omega_0 + 2 \sum_{\alpha \in S_{1/2}} e_{-\alpha} e_\alpha + 2 e_{-\theta} e_\theta,$$

hence

$$2h^\vee e_\mu = \Omega e_\mu = 2(\rho - \rho_0 | \mu) e_\mu + h_{1/2} e_\mu + 2[e_{\mu-\theta}, [e_{\theta-\mu}, e_\mu]].$$

Hence, using (5.11), we get:

$$h^\vee e_\mu = (h_{1/2} - 1)e_\mu + 2[e_{\mu-\theta}, [e_{\theta-\mu}, e_\mu]].$$

But the second summand on the right is ae_μ, where $a = c^\theta_{\theta-\mu,\mu} c^\mu_{\mu-\theta,\theta}$. Hence, it remains to show that $a = 1$. We have:

$$ae_\theta = [e_{\theta-\mu}, [e_{\mu-\theta}, e_\theta]] = [[e_{\theta-\mu}, e_{\mu-\theta}], e_\theta] = (\theta|\theta-\mu)e_\theta = e_\theta, \quad \text{hence } a = 1. \qquad \square$$

We denote by $h^\vee_{0,i}$ the dual Coxeter number of the ith simple component $\mathfrak{g}^\natural_i$ of $\mathfrak{g}^\natural$ with respect to the bilinear form $(.\,|\,.)$ restricted to $\mathfrak{g}^\natural_i$. We have the following more precise version of Theorems 2.1 and 4.1 in the case of a minimal gradation.

Theorem 5.1. (a) *All the fields* $J^{\{a\}}(z)$, $a \in \mathfrak{g}^\natural$, *and* $G^{\{v\}}(z)$, $v \in \mathfrak{g}_{-1/2}$, (*see Theorem* 2.1), *are primary fields of the vertex algebra* $W_k(\mathfrak{g}, e_{-\theta})$ *of conformal weight* 1 *and* $\frac{3}{2}$, *respectively.*

(b) *The fields* $J^{\{a\}}(z)$, $a \in \mathfrak{g}^\natural$, $G^{\{v\}}(z)$ $(v \in \mathfrak{g}_{-1/2})$ *and* $L(z)$ *strongly generate the vertex algebra* $W_k(\mathfrak{g}, e_{-\theta})$.

(c) *Let* $\{u^\alpha\}_{\alpha \in S_0}$ *be the dual basis to* $\{u_\alpha\}_{\alpha \in S_0}$, *i.e.*, $(u_\alpha|u^\beta) = \delta_{\alpha\beta}$, $\alpha, \beta \in S_0$. *Then we have the following equality in* $W_k(\mathfrak{g}, e_{-\theta})$:

$$L = -\frac{1}{k+h^\vee}\left(J^{(f)} + \sum_{\alpha \in S_{1/2}} (-1)^{p(\alpha)} \Phi^\alpha J^{([f,u_\alpha])} \right.$$
$$\left. -\frac{1}{2}\sum_{\alpha \in S_0} (-1)^{p(\alpha)} :J^{(u_\alpha)}J^{(u^\alpha)}: -\frac{1}{2}(k+h^\vee)\sum_{\alpha \in S_{1/2}} :\partial\Phi^\alpha \Phi_\alpha: -(k+1)\partial J^{(x)}\right).$$

(d) $[J^{\{a\}}{}_\lambda J^{\{b\}}] = J^{\{[a,b]\}} + \lambda\big(\big(k + \frac{1}{2}h^\vee\big)(a|b) - \frac{1}{4}\kappa_{\mathfrak{g}_0}(a,b)\big)$, *and* $[J^{\{a\}}{}_\lambda G^{\{v\}}] = G^{\{[a,v]\}}$.

(e) $$\begin{aligned}[G^{\{u\}}{}_\lambda G^{\{v\}}] = &-2(k+h^\vee)(e|[u,v])L + (e|[u,v])\sum_{\alpha \in S^\natural} :J^{\{u^\alpha\}}J^{\{u_\alpha\}}: \\ &+ \sum_{\gamma \in S_{1/2}} :J^{\{[u,u^\gamma]^\natural\}}J^{\{[u_\gamma,v]^\natural\}}: + 2(k+1)(\partial + 2\lambda)J^{\{[[e,u],v]^\natural\}} \\ &+ \lambda\sum_{\gamma \in S_{1/2}} J^{\{[[u,u^\gamma],[u_\gamma,v]]^\natural\}} + \frac{\lambda^2}{6}(e|[u,v])\Bigg(-2(k+h^\vee)c(\mathfrak{g}, e_{-\theta}, k) \\ &+ (2k+h^\vee)\,\mathrm{sdim}\,\mathfrak{g}^\natural - \sum_i h^\vee_{0,i}\,\mathrm{sdim}\,\mathfrak{g}^\natural_i \\ &+ \Big(2k + h^\vee - \sum_i h^\vee_{0,i}\Big)(2h^\vee - 3)\Bigg),\end{aligned}$$

where the superscript $\natural$ *denotes the orthogonal projection of* $\mathfrak{g}_0$ *on* $\mathfrak{g}^\natural$, u_α *and* u^α (*resp.* u_γ *and* u^γ) *are bases of* $\mathfrak{g}^\natural$ (*resp.* $\mathfrak{g}_{1/2}$) *such that* $(u_\alpha|u^{\alpha'}) = \delta_{\alpha\alpha'}$ (*resp.* $\langle u_\gamma, u^{\gamma'}\rangle_{\mathrm{ne}} = \delta_{\gamma\gamma'}$).

Proof. The claim (a) for the fields $J^{\{a\}}$ was proved in [KRW], and for the fields $G^{\{v\}}$ in Theorem 2.1(d).

In order to prove (b), it suffices to show, in view of Theorem 4.1, that L can be written in the form const. $J^{\{f\}}$ (described in Theorem 4.1(a)). From [KRW], formula (2.4), we have:

$$d_0(\varphi_\theta) = e + \frac{1}{2}, \quad d_0(f) = \sum_{\substack{\alpha\in S_+\\ \gamma\in S}} (-1)^{p(\alpha)p(\gamma)} c^\gamma_{\alpha,-\theta} :e_\gamma\varphi^\alpha: + \frac{1}{2}k\partial\varphi^\theta.$$

Since $c^\gamma_{\alpha,-\theta} \neq 0$ only if $\gamma < \theta$, we see that $:ef: + :fe:$ is homologous to $-f + \cdots$. Here and below $\cdots$ signify a linear combination of normal ordered products of fields of conformal weight <2. Hence, by the Sugawara construction, $L^{\mathfrak{g}} := \frac{1}{k+h^\vee}(:ef: + :fe: + \cdots)$ is homologous to $-\frac{1}{k+h^\vee}f + \cdots$. In view of Theorem 4.1, this proves (b). The statement (c) is obtained by solving the system of equations (4.12) for $a = -\frac{1}{k+h^\vee}f$ and inserting the solution in (4.11). We use Lemma 5.1 in this calculation. (d) follows from Theorem 2.1(c). (e) is obtained by a very long, but a straightforward λ-bracket calculation. $\square$

Remark 5.1. Let $u\in\mathfrak{g}_j$, $v\in\mathfrak{g}_{-j}$. Then we have

$$\mathrm{str}_{\mathfrak{g}_-}(\mathrm{ad}\,u)(\mathrm{ad}\,v) = \mathrm{str}_{\mathfrak{g}_+}(\mathrm{ad}\,u)(\mathrm{ad}\,v) - \mathrm{str}_{\mathfrak{g}_+}\,\mathrm{ad}\,[u,v], \tag{5.12}$$

$$\mathrm{str}_{\mathfrak{g}_+}(\mathrm{ad}\,u)(\mathrm{ad}\,v) = \tfrac{1}{2}(\kappa_{\mathfrak{g}}(u,v) - \kappa_{\mathfrak{g}_0}(u,v) + \mathrm{str}_{\mathfrak{g}_+}\,\mathrm{ad}\,[u,v]). \tag{5.13}$$

Formula (5.12) follows by a simple computation from the formulas $\mathrm{str}_{\mathfrak{g}_+} A_+ = \sum_{\alpha\in S_+}(-1)^{p(\alpha)}(A_+u_\alpha|u^\alpha)$, $\mathrm{str}_{\mathfrak{g}_-} A_- = \sum_{\alpha\in S_+}(-1)^{p(\alpha)}(u_\alpha|A_-u^\alpha)$, where $A_\pm\in\mathrm{End}\,\mathfrak{g}_\pm$, and u_α, u^α are dual bases of $\mathfrak{g}_+$ and $\mathfrak{g}_-$: $(u_\alpha|u^\beta) = \delta_{\alpha\beta}$. Formula (5.13) is immediate from (5.12). Likewise, we have $\mathrm{str}_{\mathfrak{g}_0}(\mathrm{ad}\,v)(\mathrm{ad}\,u) = (\Omega_0 v|u)$. Hence

$$\mathrm{str}_{\mathfrak{g}_0}(\mathrm{ad}\,v)(\mathrm{ad}\,u) = h_{1/2}(v|u) \quad \text{if } u\in\mathfrak{g}_{1/2}, v\in\mathfrak{g}_{-1/2}. \tag{5.14}$$

Since $[v,u] = [v,u]^\natural\text{-}(v|u)x$, $u\in\mathfrak{g}_{1/2}$, $v\in\mathfrak{g}_{-1/2}$, where the superscript $\natural$ denotes the orthogonal projection of $\mathfrak{g}_0$ on $\mathfrak{g}^\natural$, we obtain, using (5.6):

$$\mathrm{str}_{\mathfrak{g}_+}\,\mathrm{ad}\,[v,u] = -(v|u)(h^\vee - 1). \tag{5.15}$$

Table 4

$\mathfrak{g}$	$s\ell(m\|n)$ $m\neq n$	$s\ell(m\|m)/\mathbb{C}I$	$spo(n\|m)$ $(n,m)\neq(2,4)$	$osp(4\|n)$	$osp(m\|n)$ $m\geqslant 5$	$F(4)$	$G(3)$	$F(4)$	$G(3)$
$\mathfrak{g}_i^\natural$	$s\ell(m-2\|n)$	$s\ell(m-2\|m)$	$spo(n-2\|m)$	$s\ell_2$ sp_m	$s\ell_2$ $osp(m-4\|n)$	so_7	G_2	$D(2,1;2)$	$osp(3\|2)$
$h_{0,i}^\vee$	$m-n-2$	-2	$(n-m)/2$	2 $-2-n$	2 $m-n-6$	$-10/3$	-3	0	$-2/3$

It follows from (5.13), (5.14), (5.15) and Lemma 5.1 that in the case of $W_k(\mathfrak{g}, e_{-\theta})$ the last term in the formula for $G^{\{v\}}$ in Theorem 2.1 is equal to

$$-(k+1)\sum_{\beta\in S_{1/2}}(v|u_\beta)\partial\Phi^\beta = 2(-1)^{p(v)}(k+1)\partial\Phi_{[e,v]}.$$

Remark 5.2. If $[\mathfrak{g}^\natural,\mathfrak{g}^\natural]$ is simple, denoting by $h_0^\vee$ its dual Coxeter number, we have the following relation: $2+h_{1/2}\,\mathrm{sdim}\,\mathfrak{g}_{1/2} = h^\vee\,\mathrm{sdim}\,\mathfrak{g}_0 - h_0^\vee\,\mathrm{sdim}\,[\mathfrak{g}^\natural,\mathfrak{g}^\natural]$, which is obtained by calculating $\mathrm{str}_{\mathfrak{g}_+}\,\Omega_0$ using (5.12). This relation allows one to compute $h_0^\vee$. In turns out that the dual Coxeter numbers $h_{0,i}^\vee$ for $\mathfrak{g}$ from Table 1 are equal to the usual Coxeter numbers of the Lie algebras $\mathfrak{g}_i^\natural$ (as in Table 1), except for $\mathfrak{g}=G_2$, in which case $\mathfrak{g}^\natural = s\ell_2$ and $h_0^\vee = 2/3$. For Tables 2 and 3 the numbers $h_{0,i}^\vee$ are given in Table 4.

Theorem 5.1 implies a "free field realization" of all vertex algebras $W_k(\mathfrak{g}, e_{-\theta})$. In order to state the result, recall the 2-cocycle α_k on $\mathfrak{g}_0[t,t^{-1}]$ (cf. (2.5) and (5.5)):

$$\alpha_k(at^m, bt^n) = m\delta_{m,-n}((k+h^\vee)(a|b) - \tfrac{1}{2}\kappa_{\mathfrak{g}_0}(a,b)). \tag{5.16}$$

Let $\hat{\mathfrak{g}}_0 = \mathfrak{g}_0[t,t^{-1}] + \mathbb{C}1$ be the affine Lie superalgebra corresponding to this cocycle $(a,b\in\mathfrak{g}_0, m,n\in\mathbb{Z})$:

$$[at^m, bt^n] = [a,b]t^{m+n} + \alpha_k(at^m, bt^n)1,$$

and let $V_{\alpha_k}(\mathfrak{g}_0)$ be the corresponding universal affine vertex algebra.

Theorem 5.2. *The following formulas define a vertex algebra homomorphism of $W_k(\mathfrak{g}, e_{-\theta})$ to $V_{\alpha_k}(\mathfrak{g}_0)\otimes F(A_{\mathrm{ne}})$:*

$$J^{\{a\}} \mapsto a + \frac{(-1)^{p(a)}}{2}\sum_{\alpha\in S_{1/2}} :\Phi^\alpha\Phi_{[u_\alpha,a]}: (a\in\mathfrak{g}^\natural),$$

$$G^{\{v\}} \mapsto \sum_{\alpha \in S_{1/2}} :[v, u_\alpha]\Phi^\alpha: -(k+1) \sum_{\alpha \in S_{1/2}} (v|u_\alpha)\partial\Phi^\alpha - \frac{(-1)^{p(v)}}{3} \sum_{\alpha,\beta \in S_{1/2}} :\Phi^\alpha\Phi^\beta\Phi_{[u_\beta,[u_\alpha,v]]}: \ (v \in \mathfrak{g}_{-1/2}),$$

$$L \mapsto \frac{1}{2(k+h^\vee)} \sum_{\alpha \in S_0} (-1)^{p(\alpha)} :u_\alpha u^\alpha: + \frac{k+1}{k+h^\vee}\partial x + \frac{1}{2} \sum_{\alpha \in S_{1/2}} :\partial\Phi^\alpha\Phi_\alpha: .$$

Proof. Let $\hat{\mathfrak{g}}_\leqslant$ be the affine superalgebra associated to the Lie superalgebra $\mathfrak{g}_\leqslant$ and the 2-cocycle α_k on $\mathfrak{g}_\leqslant[t, t^{-1}]$ extended from $\mathfrak{g}_0[t, t^{-1}]$ to $\mathfrak{g}_\leqslant[t, t^{-1}]$ trivially. Theorems 2.1 and 5.1 give us a realization of the vertex algebra $W_k(\mathfrak{g}, e_{-\theta})$ as a subalgebra of the vertex algebra $V_{\alpha_k}(\mathfrak{g}_\leqslant) \otimes F(A_{\mathrm{ne}})$. The realization given by Theorem 5.2 is given by the homomorphism of $V_{\alpha_k}(\mathfrak{g}_\leqslant) \otimes F(A_{\mathrm{ne}})$ to $V_{\alpha_k}(\mathfrak{g}_0) \otimes F(A_{\mathrm{ne}})$ induced by the canonical homomorphism $\mathfrak{g}_\leqslant \to \mathfrak{g}_0$. (We use also Remark 5.1 to simplify a coefficient in $G^{\{v\}}$.) $\square$

Remark 5.3. Replacing $V_{\alpha_k}(\mathfrak{g}_0)$ and $F(A_{\mathrm{ne}})$ in Theorem 5.2 by their twisted versions, we obtain free field realizations of twisted superconformal algebras, in particular, the Ramond sector (see [KW5] for details).

6. Highest weight modules over $W_k(\mathfrak{g}, x, f)$

Let V be a vertex algebra with a conformal vector v, that is the field $Y(v, z)$, corresponding to v, is of the form $Y(v, z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}$, where the L_n form a representation of the Virasoro algebra with central charge c, L_{-1} coincides with the infinitesimal translation operator and L_0 is diagonalizable on V [K4]. If $a \in V$ is an eigenvector of L_0 with eigenvalue = conformal weight $\Delta(a)$, one writes $Y(a, z) = \sum_{n \in \mathbb{Z} - \Delta(a)} a_n z^{-n-\Delta(a)}$. Recall [K4] that one has the following commutator formula for $Y(a, z)$ and $Y(b, w)$:

$$[a_m, b_n] = \sum_{j \in \mathbb{Z}_+} \binom{\Delta(a) + m - 1}{j} (a_{(j)}b)_{m+n}, \tag{6.1}$$

where $\Delta(a)$ is the conformal weight of a and $a_{(j)}b$ is the jth product in V. In particular,

$$[L_0, a_n] = -na_n. \tag{6.2}$$

Recall that a V-module is a vector space M and a collection of End M-valued fields $\{Y^M(a, z) = \sum_{n \in \mathbb{Z} - \Delta_{(a)}} a_n^M z^{-n-\Delta_{(a)}}\}_{a \in V}$, such that the following properties hold:

(M1) $Y^M(|0\rangle, z) = I_M$, i.e., $|0\rangle_n^M = \delta_{n,0}$,
(M2) $Y^M(L_{-1}a, z) = \frac{d}{dz} Y^M(a, z)$, i.e., $(L_{-1}a)_n^M = -(n + \Delta(a))a_n^M$,

(M3) $Y^M(a_{(-1)}b, z) =: Y^M(a, z)Y^M(b, z) :$,
(M4) $[a^M_m, b^M_m] = \sum_{j\in\mathbb{Z}_+} \binom{\Delta(a)+m-1}{j}(a_{(j)}b)^M_{m+n}$ (cf. (6.1)),
(M5) L^M_0 is diagonalizable on M with eigenvalues bounded below.

Suppose that V is strongly generated by a collection of fields $\{J^{\{i\}}(z) = \sum_{m\in\mathbb{Z}-\Delta(i)} J^{\{i\}}_m z^{-m-\Delta(i)}\}_{i\in I}$, where $J^{\{i\}}$ has conformal weight $\Delta(i)\in\mathbb{R}$. Then by (6.1), we have:

$$[J^{\{i\}}_m, J^{\{j\}}_n] = \sum_{\vec{s},\vec{t}} c^{ij}_{m,n}(\vec{s},\vec{t})J^{\{s_1\}}_{t_1}J^{\{s_2\}}_{t_2}\dots, \tag{6.3}$$

where the sum is finite and for each term of this sum we have:

$$t_r\in\mathbb{Z} - \Delta(s_r), \sum_r t_r = m+n \text{ and } t_1\leqslant t_2\leqslant\cdots. \tag{6.4}$$

Denote by $\mathscr{A}$ the unital associative superalgebra on generators $J^{\{i\}}_m$ $(i\in I, m\in\mathbb{Z} - \Delta(i))$ and defining relations (6.3). Let $\tilde{\mathscr{A}}_-$, $\tilde{\mathscr{A}}_+$ and $\tilde{\mathscr{A}}_0$ be the subalgebras of $\mathscr{A}$ generated by the $J^{\{i\}}_m$ with $m<0$, $m>0$ and $m=0$, respectively. It follows from (M4) that any V-module M is an $\mathscr{A}$-module. It follows from (6.3) and (6.4) that

$$\mathscr{A} = \tilde{\mathscr{A}}_-\tilde{\mathscr{A}}_0\tilde{\mathscr{A}}_+. \tag{6.5}$$

By (6.2), L_0 lies in the center of $\tilde{\mathscr{A}}_0$, hence each eigenspace of L_0 in M carries a representation of $\tilde{\mathscr{A}}_0$.

Denote by $\mathscr{A}_0$ the unital associative superalgebra on generators $J^{\{i\}}_0$ $(i\in I)$ and the defining relations (cf. (6.3)):

$$[J^{\{i\}}_0, J^{\{j\}}_0] = \sum_{\vec{s}} c^{ij}_{00}(\vec{s}, 0)J^{\{s_1\}}_0 J^{\{s_2\}}_0\dots. \tag{6.6}$$

It is clear from (6.3), (6.4) and (6.5) that the representation of $\tilde{\mathscr{A}}_0$ in the eigenspace of L^M_0 with the minimal eigenvalue induces a representation of the associative superalgebra $\mathscr{A}_0$.

The proof of the following theorem is immediate from the above remarks.

Theorem 6.1. *Let M be an irreducible V-module. Then the representation of $\mathscr{A}_0$ in the eigenspace of L^M_0 with the minimal eigenvalue is irreducible, and this representation uniquely determines M.*

Example 6.1. Let $V = W_k(\mathfrak{g}, e_{-\theta})$. Recall (Theorem 5.1) that V is strongly generated by L, primary fields $J^{\{a\}}$ $(a\in\mathfrak{g}^\natural)$ of conformal weight 1 and primary fields $G^{\{v\}}$ $(v\in\mathfrak{g}_{-1/2})$ of conformal weight $3/2$. The fields $G^{\{v\}}$ obviously do not contribute to $\mathscr{A}_0$, and, by Theorem 2.1(c), relation (6.6) is simply $[J_0^{\{a\}}, J_0^{\{b\}}] = J_0^{\{[a,b]\}}$. Hence

$$\mathscr{A}_0 = U(\mathfrak{g}^\natural) \otimes \mathbb{C}[L_0] \simeq U(\mathfrak{g}_0).$$

Recall that, given a restricted $\hat{\mathfrak{g}}$-module P of level k (i.e., $K = kI_P$), one constructs the associated $\mathscr{C}(\mathfrak{g}, x, f, k)$-module

$$\mathscr{C}(P) = P \otimes F(\mathfrak{g}, x, f)$$

with the differential $d_0^P = \mathrm{Res}_z\, d^{\mathscr{C}(P)}(z)$. The homology $H(P)$ of the complex $(\mathscr{C}(P), d_0^P)$ with the induced charge decomposition (by setting charge $P = 0$) is a direct sum of $W_k(\mathfrak{g}, x, f)$-modules:

$$H(P) = \bigoplus_{j \in \mathbb{Z}} H_j(P).$$

Thus we get a functor from the category of restricted $\hat{\mathfrak{g}}$-modules to the category of $\mathbb{Z}$-graded $W_k(\mathfrak{g}, x, f)$-modules [FF2,FKW,KRW].

Let P_0 be a $\mathfrak{g}$-module on which x is diagonalizable and $U(\mathfrak{g}_+)$ is locally finite (for example, a Verma module over $\mathfrak{g}$). Extend P_0 to a $\mathfrak{g}[t] + \mathbb{C}D + \mathbb{C}K$-module by letting D and $\mathfrak{g}t^n$ for $n \geqslant 1$ act trivially, and $K = kI_{P_0}$. The induced $\hat{\mathfrak{g}}$-module

$$P = U(\hat{\mathfrak{g}}) \otimes_{U(\mathfrak{g}[t] + \mathbb{C}D + \mathbb{C}K)} P_0 \tag{6.7}$$

is called a *generalized Verma* module over $\hat{\mathfrak{g}}$. (For example, if P_0 is a Verma module over $\mathfrak{g}$, then P is a Verma module over $\hat{\mathfrak{g}}$.)

Theorem 6.2. *Suppose that* (x, f) *is a good pair* (*i.e.*, (1.10) *holds*) *and that* P *is a generalized Verma module over* $\hat{\mathfrak{g}}$. *Then*

$$H_j(P) = 0 \quad \textit{if } j \neq 0.$$

Proof. We have:

$$\mathscr{C}(P) = \mathscr{C}(\mathfrak{g}, x, f, k) \otimes P_0$$

with the following action of the differential:

$$d_0^P(a \otimes v) = d_0(a) \otimes v + \left(\sum_{n \in \mathbb{Z}} \sum_{\alpha \in S_+} (-1)^{p(\alpha)} \varphi^{\alpha}_{(n-1)} \otimes u_{\alpha(-n)} \right) (a \otimes v), \tag{6.8}$$

where $a \in \mathscr{C}(\mathfrak{g}, x, f, k)$, $v \in P_0$.

Recall that we have decomposition (4.2) of the complex $\mathscr{C}(\mathfrak{g}, x, f, k)$, hence

$$(\mathscr{C}(P), d_0^P) = (\mathscr{C}^+, d_0) \otimes (\mathscr{C}^- \otimes P_0, d_0^P)$$

is a decomposition of complexes. Hence by (4.1) and the Künneth lemma, we need to prove

$$H_j(\mathscr{C}^- \otimes P_0, d_0^P) = 0 \quad \text{if } j \neq 0. \tag{6.9}$$

Recall that in Section 4 we introduced a bicomplex structure on $(\mathscr{C}^-, d_0)$:

$$\mathscr{C}^- = \bigoplus_{m,n \in \frac{1}{2}\mathbb{Z}} \mathscr{C}^-_{m,n}, \quad d_0 = d_1 + d_2.$$

We extend this bicomplex structure to the complex $(\mathscr{C}^- \otimes P_0, d_0^P)$ by letting

$P_0 = \bigotimes_{j \in \frac{1}{2}\mathbb{Z}} (P_0)_{j,-j}$, where $(P_0)_{j,-j}$ is the eigenspace of x with the eigenvalue j,

and letting

$$d_1^P = d_1 \otimes 1, \quad d_2^P = d_2 \otimes 1 + \sum_{n \in \mathbb{Z}} \sum_{\alpha \in S_+} (-1)^{p(\alpha)} \varphi^\alpha_{(n-1)} \otimes u_{\alpha(-n)}.$$

Since $d_1(\varphi^\alpha) = 0$, due to Lemma 4.1, we have: $d_1^P d_2^P + d_2^P d_1^P = 0$. Also, obviously $(d_1^P)^2 = 0$, and since $(d_0^P)^2 = 0$, we conclude that $(d_2^P)^2 = 0$.

But by Lemma 4.2, $H_j(\mathscr{C}^- \otimes P_0, d_1^P) = 0$ if $j \neq 0$. Also the bicomplex $\mathscr{C}^- \otimes P_0$ is locally finite since P_0 is a locally finite $U(\mathfrak{g}_+)$-module. Hence the spectral sequence converges to the homology of $(\mathscr{C}^- \otimes P, d_0^P)$ and (6.9) holds. $\square$

Let $\mathfrak{h}^\natural$ be a maximal diagonalizable subalgebra of the even part of $\mathfrak{g}^\natural$, and include $\mathfrak{h}^\natural$ in a Cartan subalgebra $\mathfrak{h}$ of the even part of $\mathfrak{g}_0$. Let $\Delta_j \subset \mathfrak{h}^*$ be the set of roots of $\mathfrak{h}$ in $\mathfrak{g}_j$ and let Δ_{0+} be a subset of positive roots of Δ_0. Then $\mathfrak{h}$ is a Cartan subalgebra of the even part of $\mathfrak{g}$ and $\Delta = \coprod_j \Delta_j$ is the set of roots of $\mathfrak{h}$ in $\mathfrak{g}$, $\Delta_+ := \Delta_{0+} \coprod (\coprod_{j>0} \Delta_j)$ being the subset of positive roots. Denoting by $\mathfrak{n}_{0+}$, $\mathfrak{n}_{0-}$, $\mathfrak{n}_+$ and $\mathfrak{n}_-$ the sum of all root spaces corresponding to Δ_{0+}, $-\Delta_{0+}$, Δ_+ and $-\Delta_+$, we obtain the triangular decompositions:

$$\mathfrak{g} = \mathfrak{n}_0 \oplus \mathfrak{h} \oplus \mathfrak{n}_+ \quad \mathfrak{g}_0 = \mathfrak{n}_{0-} \oplus \mathfrak{h} \oplus \mathfrak{n}_{0+}. \tag{6.10}$$

Recall that a Verma module over $\mathfrak{g}$ with highest weight $\lambda \in \mathfrak{h}^*$ is the module $U(\mathfrak{g}) \otimes_{U(\mathfrak{h}+\mathfrak{n}_+)} \mathbb{C}v_\lambda$, where $\mathbb{C}v_\lambda$ is a 1-dimensional $\mathfrak{h} + \mathfrak{n}_+$-module such that $\mathfrak{n}_+ v_\lambda = 0$ and $hv_\lambda = \lambda(h)v_\lambda$ for all $h \in \mathfrak{h}$.

Let $\hat{\mathfrak{h}} = \mathfrak{h} + \mathbb{C}K + \mathbb{C}D$ be the Cartan subalgebra of $\hat{\mathfrak{g}}$. We extend the bilinear form $(.|.)$ from $\mathfrak{h}$ to $\hat{\mathfrak{h}}$ by letting $(\mathfrak{h}|\mathbb{C}K + \mathbb{C}D) = 0$, $(D|D) = (K|K) = 0$, $(K|D) = (D|K) = 1$, and identify $\hat{\mathfrak{h}}$ with $\hat{\mathfrak{h}}^*$ using this form. Given k, we extend $\lambda \in \mathfrak{h}^*$ to $\hat{\lambda} \in \hat{\mathfrak{h}}^*$ letting $\hat{\lambda}|_{\mathfrak{h}} = \lambda$, $\lambda(K) = k$, $\lambda(D) = 0$.

A Verma module over $\hat{\mathfrak{g}}$ with highest weight $\hat{\lambda}$ of level k is a generalized Verma module (6.7) for which P_0 is a Verma $\mathfrak{g}$-module with highest weight λ. Any quotient of a Verma $\hat{\mathfrak{g}}$-module with highest weight $\hat{\lambda}$ is called a highest weight $\hat{\mathfrak{g}}$-module.

Recall that the Euler–Poincaré character of the $\mathbb{Z}$-graded $W_k(\mathfrak{g}, x, f)$-module $H(P)$ is defined by:

$$\mathrm{ch}_{H(P)}(h) = \sum_{j \in \mathbb{Z}} (-1)^j \, \mathrm{tr}_{H_j(P)} \, q^{L_0} e^{J_0^{\{h\}}}, \quad h \in \mathfrak{h}^\natural.$$

As usual, we define the character of a $\hat{\mathfrak{g}}$-module P by $\mathrm{ch}_P(H) = \mathrm{tr}_P \, e^H$, where $H \in \hat{\mathfrak{h}}$. We let, as usual, $\hat{\rho} = h^\vee D + \rho$. Let P be a highest weight $\hat{\mathfrak{g}}$-module with highest weight $\hat{\lambda}$ of level k. Due to [KRW] (see Remark 3.1 there), we have the following formula for $\mathrm{ch}_{H(P)}$ in terms of ch_P:

$$\begin{aligned}\mathrm{ch}_{H(P)}(h) = & \frac{q^{\frac{(\hat{\lambda}|\hat{\lambda}+2\hat{\rho})}{2(k+h^\vee)}}}{\prod_{j=1}^{\infty} (1-q^j)^{\dim \mathfrak{h}}} \hat{R} \, \mathrm{ch}_P(2\pi i(-\tau D - \tau x) + h) \\ & \times \prod_{n=1}^{\infty} \left(\prod_{\alpha \in \Delta_{0+}} (1 - s(\alpha) q^{n-1} e^{-\alpha(h)})^{-s(\alpha)} (1 - s(\alpha) q^n e^{\alpha(h)})^{-s(\alpha)} \right. \\ & \left. \times \prod_{\alpha \in \Delta_{1/2}} (1 - s(\alpha) q^{n-\frac{1}{2}} e^{\alpha(h)})^{-s(\alpha)} \right), \end{aligned} \tag{6.11}$$

where $\hat{R}$ is the Weyl denominator for $\hat{\mathfrak{g}}$, $s(\alpha) = (-1)^{p(\alpha)}$, $q = e^{2\pi i \tau}$, $h \in \mathfrak{h}^\natural$.

Now we shall define a Verma module over the vertex algebra $W_k(\mathfrak{g}, x, f)$, assuming that the pair (x, f) is good. The isomorphism (1.12) and the triangular decomposition (6.10) of $\mathfrak{g}_0$ induce the following decomposition of $\mathfrak{g}^f$ as $\mathfrak{h}^\natural$-modules:

$$\mathfrak{g}^f = \mathfrak{n}_{0-} \oplus \mathfrak{h} \oplus \mathfrak{n}_{0+} \oplus \mathfrak{g}_{1/2}, \tag{6.12}$$

compatible with the gradation (1.10).

Example 6.2. In the case of a minimal gradation we have $\mathfrak{g}^\natural = \mathfrak{n}_{0-} \oplus \mathfrak{h}^\natural \oplus \mathfrak{n}_{0+}$ and $\mathfrak{h} = \mathfrak{h}^\natural + \mathbb{C}x$, so that the decomposition (6.12) is identified with (5.2) via the identification of x with f and $\mathfrak{g}_{1/2}$ with $\mathfrak{g}_{-1/2}$ given by $\mathrm{ad} f$.

Choose a basis $\{a_i \mid i = 1, \dots, s\}$ of $\mathfrak{g}^f$ compatible with the decomposition (6.12), the root space decomposition with respect to $\mathfrak{h}^\natural$ of (6.12) and the gradation (1.10). Let $J^{\{a_i\}}(z) = \sum_{n \in \mathbb{Z} - \Delta(i)} J_n^{\{i\}} z^{-n-\Delta(i)}$ be the field of $W_k(\mathfrak{g}, x, f)$ corresponding to a_i (given by Theorem 4.1a). Recall that the conformal weight $\Delta(i) = 1 + j$ if $a_i \in \mathfrak{g}_{-j}$ $(j \in \frac{1}{2}\mathbb{Z}_+)$. We order the a_i in such a way that $a_1, \dots, a_r = x$ form a basis of $\mathfrak{h}$. A $W_k(\mathfrak{g}, x, f)$-module M is called a *highest weight module* with *highest weight* $\lambda = (\lambda(a_1), \dots, \lambda(a_r)) \in \mathbb{C}^r$, if there exists a non-zero vector v such that:

(HW1) $W_k(\mathfrak{g}, x, f)v = M$,

(HW2) $J_0^{\{i\}} v = \lambda(a_i) v$ if $a_i \in \mathfrak{h}$,

(HW3) $J_m^{\{i\}} v = 0$ if $m > 0$, or $m = 0$ and $a_i \in \mathfrak{n}_{0+}$.

Note that the following vectors span a highest weight $W_k(\mathfrak{g}, x, f)$-module M:

$$(J^{\{i_1\}}_{-m_1})^{b_1} \cdots (J^{\{i_s\}}_{-m_s})^{b_s} v, \text{ where } b_i \in \mathbb{Z}_+, b_i \leqslant 1 \text{ if } a_i \text{ is odd}, \; m_i > 0,$$
$$\text{or } m_i = 0 \text{ and } a_i \in \mathfrak{n}_{0-}. \tag{6.13}$$

A highest weight $W_k(\mathfrak{g}, x, f)$-module M is called a Verma module if the vectors (6.13), where the sequence of pairs (i_1, m_1), (i_2, m_2), … decreases in lexicographical order, form a basis of M (cf. Remark 4.2). It is clear that any irreducible $W_k(\mathfrak{g}, x, f)$-module is a highest weight module, hence is a quotient of the Verma module with the same highest weight.

Theorem 6.3. *Let P be a Verma $\hat{\mathfrak{g}}$-module with highest weight $\hat{\lambda} \in \hat{\mathfrak{h}}^*$, where $\hat{\lambda}|_{\mathfrak{h}} = \lambda$, $\hat{\lambda}(K) = k$, $\hat{\lambda}(D) = 0$. Then $H(P) = H_0(P)$, and it is a Verma $W_k(\mathfrak{g}, x, f)$-module with highest weight $\lambda_W \in \mathfrak{h}^*$ such that*

$$\lambda_W|_{\mathfrak{h}^\natural} = \lambda|_{\mathfrak{h}^\natural} \quad \textit{and} \quad \lambda_W(x) = \frac{(\hat{\lambda}|\hat{\lambda} + 2\hat{\rho})}{2(k + h^\vee)} - \lambda(x). \tag{6.14}$$

Proof. It is clear from (6.8) that the vector $v = v_{\hat{\lambda}} \otimes |0\rangle$, where $v_{\hat{\lambda}}$ is a highest weight vector of P and $|0\rangle$ is the vacuum vector of $F(\mathfrak{g}, x, f)$, is d_0^P-closed in $\mathscr{C}(P)$. Furthermore, $W_k(\mathfrak{g}, x, f)v$ is obviously a highest weight submodule of the $W_k(\mathfrak{g}, x, f)$-module $H(P)$.

Next, it is clear from Theorem 4.1 that $W_k(\mathfrak{g}, x, f)v$ is a Verma module. Due to Theorem 6.2, it remains to show that the Euler–Poincaré character of $H(P)$ coincides with the character of a $W_k(\mathfrak{g}, x, f)$-Verma module. Since $\hat{R}\, \mathrm{ch}_P = e^{\hat{\lambda}}$ by definition of a Verma $\hat{\mathfrak{g}}$-module P with highest weight $\hat{\lambda}$, we obtain from (6.11) the following formula ($h \in \mathfrak{h}^\natural$):

$$\begin{aligned}
\mathrm{ch}_{H(P)}(h) &= e^{\lambda(h)} q^{\frac{(\hat{\lambda}|\hat{\lambda}+2\hat{\rho})}{2(k+h^\vee)} - (\lambda|x)} \prod_{j=1}^{\infty} (1 - q^j)^{-\dim \mathfrak{h}} \\
&\quad \times \prod_{n=1}^{\infty} \Bigg(\prod_{\alpha \in \Delta_{0+}} (1 - s(\alpha) q^{n-1} e^{-\alpha(h)})^{-s(\alpha)} (1 - s(\alpha) q^n e^{\alpha(h)})^{-s(\alpha)} \\
&\quad \times \prod_{\alpha \in \Delta_{1/2}} (1 - s(\alpha) q^{n-\frac{1}{2}} e^{\alpha(h)})^{-s(\alpha)} \Bigg).
\end{aligned}$$

But, in view of (6.13), this is precisely the character of a $W_k(\mathfrak{g}, x, f)$-Verma module with highest weight λ_W satisfying (6.14). $\square$

Corollary 6.1. (a) *For any* $\lambda \in \mathbb{C}^r$ *there exists a Verma* $W_k(\mathfrak{g}, x, f)$*-module with highest weight* λ.

(b) *If* $a_i, a_j \in \mathfrak{h}$, *then* $[J_0^{\{i\}}, J_0^{\{j\}}] = 0$ *in* $\mathscr{A}_0$.

Proof. (a) is immediate by Theorem 6.3. (b) follows from (a) and (6.6) by induction on the conformal weight of $J^{\{a_i\}}$, using that $J_0^{\{i\}} v_\lambda = \lambda(a_i) v$ if v is the highest weight vector (cf. [FKW]). $\square$

In conclusion of this section, we construct an anti-involution ω of the associative superalgebra $\mathscr{A}$ and the corresponding contravariant form on any highest weight module over $W_k(\mathfrak{g}, x, f)$. We call an *anti-involution* of an associative superalgebra (resp. Lie superalgebra) a vector superspace automorphism ω such that $\omega^2 = 1$ and $\omega(ab) = \omega(b)\omega(a)$ (resp. $\omega[a,b] = [\omega(b), \omega(a)]$).

First, we construct an anti-involution ω of the Lie superalgebra $\mathfrak{g}$ such that the following properties hold: $\omega(x) = -x$, $\omega(a) = a$ if $a \in \mathfrak{h}^\natural$, $\omega(f) = f$, $\omega(\mathfrak{g}_\alpha) = \mathfrak{g}_{-\alpha}$ if $\alpha \in \Delta_0$, $(\omega(a)|\omega(b)) = (b|a)$, $a, b \in \mathfrak{g}$.

Next, we lift ω to an anti-involution of the affine Lie superalgebra $\hat{\mathfrak{g}}$ by letting $\omega(a_n) = \omega(a)_{-n} + 2K(a|x)\delta_{0,n}$ (the extra term occurs because the field $x(z)$ is not primary), $\omega(K) = K$, $\omega(D) = D$. Recall that we write $a(z) = \sum_{n \in \mathbb{Z} - \Delta} a_n z^{-n-\Delta}$, where Δ is the conformal weight of $a(z)$. Furthermore, ω induces a super vector space automorphism of A_{ne} such that $\langle \omega(a), \omega(b) \rangle_{\mathrm{ne}} = \langle b, a \rangle_{\mathrm{ne}}$. Also ω induces a super vector space automorphism of A, which in turn induces an automorphism of A^* by $\omega(b^*)(a) = -(-1)^{p(a)p(b)} b^*(\omega(a))$, so that we have on A_{ch}: $\langle \omega(a), \omega(b) \rangle_{\mathrm{ch}} = \langle b, a \rangle_{\mathrm{ch}}$. We extend this ω to an anti-involution of $\hat{A}_{\mathrm{ne}}$ (resp. $\hat{A}_{\mathrm{ch}}$) by letting $\omega(\Phi_n) = \omega(\Phi)_{-n}$, $\Phi \in A_{\mathrm{ne}}$ (resp. $\omega(\varphi_n) = -(-1)^{p(\varphi)} \omega(\varphi)_{-n}$, $\omega(\varphi_n^*) = \omega(\varphi^*)_{-n}$, $\varphi \in A$, $\varphi^* \in A^*$), $\omega(K) = K$. Next, ω induces an anti-automorphism of the tensor product of $U_k(\hat{\mathfrak{g}}) = U(\hat{\mathfrak{g}})/(K-k)$, $U_1(\hat{A}_{\mathrm{ne}})/(K-1)$ and $U_1(\hat{A}_{\mathrm{ch}})/(K-1)$. The associative superalgebra $\mathscr{A}(\mathscr{C})$ corresponding to the vertex algebra $\mathscr{C}(\mathfrak{g}, x, f, k)$ is the "local completion" of this tensor product, i.e., it is generated by the Fourier coefficients of normal ordered products of currents $a(z)$, $a \in \mathfrak{g}$, ghosts $\varphi_\alpha(z)$, $\varphi^\alpha(z)$, $\Phi_\alpha(z)$ and their derivatives. We extend the anti-involution ω to $\mathscr{A}(\mathscr{C})$ in the obvious way.

In order to compute the action of ω on normally ordered products, we need the following lemma.

Lemma 6.1. *Let U be an associative superalgebra and let ω be its anti-involution. Let* $a(z) = \sum_{n \in \mathbb{Z} - \Delta_a} a_n z^{-n-\Delta_a}$, $b(z) = \sum_{n \in \mathbb{Z} - \Delta_b} b_n z^{-n-\Delta_b}$ *be two U-valued formal distributions such that* $\Delta_a, \Delta_b \in \frac{1}{2}\mathbb{Z}$, *and let* $:ab:(z) = \sum_{n \in \mathbb{Z} - \Delta_a - \Delta_b} :ab:_n z^{-n-\Delta_a-\Delta_b}$ *be their normally ordered product. Define the formal distribution $\omega(a)$ (and similarly $\omega(b)$) by* $\omega(a)(z) = \sum_{n \in -\Delta_a + \mathbb{Z}} \omega(a_{-n}) z^{-n-\Delta_a}$. *Then*:

$$\omega(:ab:_n) =: \omega(b)\omega(a) :_{-n} + \sum_{k \in \mathbb{Z}_+} \binom{-n - \Delta_a + \Delta_b}{k+1} (\omega(b)_{(k)}\omega(a))_{-n}.$$

Proof. We have for $n \in -\Delta_a - \Delta_b + \mathbb{Z}$:

$$:ab:_n = \sum_{k \in -\Delta_a - \mathbb{Z}_+} a_k b_{n-k} + (-1)^{p(a)p(b)} \sum_{k \in -\Delta_a + 1 + \mathbb{Z}_+} b_{n-k} a_k.$$

We let $n_{ab} = -n - \Delta_a + \Delta_b$ to simplify notation (note that it is an integer). Applying ω to both sides and replacing k by $j + n$, where $j \in -\Delta_b + \mathbb{Z}$, we obtain:

$$\omega(:ab:_n) = \sum_{j+\Delta_b \leqslant n_{ab}} \omega(b)_j \omega(a)_{-j-n} + (-1)^{p(a)p(b)} \sum_{j+\Delta_b > n_{ab}} \omega(a)_{-j-n} \omega(b)_j.$$

First, consider the case $n_{ab} < 0$. Then the last equality can be rewritten as follows:

$$\omega(:ab:_n) = \omega(b)\omega(a):_{-n} - \sum_{n_{ab} < j+\Delta_b \leqslant 0} [\omega(b)_j, \omega(a)_{-j-n}].$$

The commutator formula (6.1) gives the result in this case, by making use of the following identity that holds for any negative integer A and non-negative integer k: $\sum_{s=A}^{-1} \binom{s}{k} = -\binom{A}{k+1}$. The case $n_{ab} \geqslant 0$ is treated similarly. $\square$

Using Lemma 6.1, it is straightforward to check the following formulas:

$$\omega(L_n) = L_{-n}, \tag{6.15}$$

$$\omega(v_n^{\mathrm{ch}}) = \omega(v)_{-n}^{\mathrm{ch}} + \delta_{j0}\delta_{n0}(2h^\vee (x|v) - \mathrm{str}_{\mathfrak{g}_+} \mathrm{ad}\, v) \quad \text{if } v \in \mathfrak{g}_j, \tag{6.16}$$

$$\omega(v_n^{ne}) = (\omega v)_{-n}^{\mathrm{ne}} \quad \text{if } v \in \mathfrak{g}^\natural, \tag{6.17}$$

$$\omega(J_n^{(v)}) = J_{-n}^{(\omega(v))} + \delta_{j0}\delta_{n0}(2(k+h^\vee)(x|v) - \mathrm{str}_{\mathfrak{g}_+} \mathrm{ad}\, v) \quad \text{if } v \in \mathfrak{g}_j, \tag{6.18}$$

$$\omega(J_n^{\{v\}}) = J_{-n}^{\{\omega(v)\}} \quad \text{if } v \in \mathfrak{g}^\natural. \tag{6.19}$$

It is also straightforward to check that $\omega(d_n) = d_{-n}$, so that d_0 is fixed by ω. Hence ω induces an anti-involution of the associative superalgebra $\mathscr{A}$ corresponding to the vertex algebra $W_k(\mathfrak{g}, x, f)$, which we again denote by ω. It follows from (6.18) that, for $v \in \mathfrak{g}_{-1/2}^f$, we have: $\omega(G^{\{v\}}) = J^{(\omega(v))} + \cdots$, where $\cdots$ denote a field from the subalgebra strongly generated by the $J^{(a)}$, $a \in \mathfrak{g}_0$, and the Φ_α. Since the fields $G^{\{\omega(v)\}}$ and $\omega(G^{\{v\}})$ are d_0-closed, it follows from Remark 4.1 that $\omega(G^{\{v\}}) = G^{\{\omega(v)\}}$, hence we obtain

$$\omega(G_n^{\{v\}}) = G_{-n}^{\{\omega(v)\}} \quad \text{if } v \in \mathfrak{g}_{-1/2}^f. \tag{6.20}$$

Now let M be a Verma module over $W_k(\mathfrak{g}, x, f)$ with highest weight vector v_λ. Any vector $v \in M$ can be written uniquely in the form $v = \langle v \rangle v_\lambda + v'$, where $\langle v \rangle \in \mathbb{C}$ is called the *expectation value* of v, and v' is a linear combination of weight vectors v_μ with $\mu \neq \lambda$. The basic property of the expectation value, which follows from decomposition (6.5) is

$$\langle \omega(a) v_\lambda \rangle = \langle a v_\lambda \rangle, \quad a \in \mathscr{A}. \tag{6.21}$$

We define the *contravariant bilinear form* $B(.\,,\,.)$ on M by the formula:

$$B(a v_\lambda, b v_\lambda) = \langle \omega(a) b v_\lambda \rangle, \quad \text{where } a, b \in \mathscr{A}.$$

By the definition, this bilinear form is contravariant:

$$B(au, v) = B(u, \omega(a)v), \quad u, v \in M, a \in \mathscr{A}, \tag{6.22}$$

we have

$$B(v_\lambda, v_\lambda) = 1, \tag{6.23}$$

and it follows from (6.21) that it is symmetric:

$$B(u, v) = B(v, u), \quad u, v \in M. \tag{6.24}$$

Properties (6.22)–(6.24) determine the form $B(.\,,\,.)$ uniquely.

As usual the maximal submodule of the Verma module M coincides with the kernel of the form $B(.\,,\,.)$.

7. Highest weight modules over vertex algebras $W_k(\mathfrak{g}, e_{-\theta})$ and the determinant formula

Theorem 7.1. (a) *The map that associates to an irreducible $W_k(\mathfrak{g}, e_{-\theta})$-module M the lowest eigenvalue h of L_0 and the representation of the Lie superalgebra $\mathfrak{g}^\natural$ in the corresponding eigenspace M_h, given by $a \mapsto J_0^{\{a\}}$, $a \in \mathfrak{g}^\natural$, is a bijection between irreducible modules of $W_k(\mathfrak{g}, e_{-\theta})$ and the set of pairs (h, π), where π is an irreducible representation of $\mathfrak{g}^\natural$.*

(b) *All Verma $W_k(\mathfrak{g}, e_{-\theta})$-modules are of the form $H_0(P)$, where P is a Verma $\hat{\mathfrak{g}}$-module with highest weight $\hat{\lambda}$. The highest weight λ_W of such a $W_k(\mathfrak{g}, e_{-\theta})$-Verma module is completely determined by* (6.14), *the number $\lambda_W(x)$ being the lowest eigenvalue of L_0.*

(c) *The Euler–Poincaré character* $\mathrm{ch}_{H(P)}$ *is non-zero if and only if* $t^{-1}e_\theta$ *is not locally nilpotent in the* $\hat{\mathfrak{g}}$*-module* P.

Proof. (a) and (b) follow from Theorems 6.1, 6.3, Example 6.1 and (6.3). (c) follows from [KRW], Theorem 3.2. □

Consider now a Verma module M over the vertex algebra $W_k(\mathfrak{g}, e_{-\theta})$. Its highest weight Λ_W is a pair Λ, h, where $\Lambda \in \mathfrak{h}^{\natural *}$ and $h \in \mathbb{C}$ is the lowest (i.e., $\mathrm{Re}\, h$ is minimal) eigenvalue of L_0. We have:

$$J_0^{\{a\}} v_{(\Lambda,h)} = \Lambda(a) v_{(\Lambda,h)}, \quad a \in \mathfrak{h}^\natural, \quad L_0 v_{(\Lambda,h)} = h v_{(\Lambda,h)},$$

where $v_{(\Lambda,h)}$ is the highest weight vector of M. With respect to the commuting pair $(\mathfrak{h}^\natural, L_0)$, M decomposes into a direct sum of weight spaces:

$$M = \bigoplus_{\substack{\lambda \in \mathfrak{h}^{\natural *} \\ m \in \frac{1}{2}\mathbb{Z}_+}} M_{(\lambda, h+m)}, \tag{7.1}$$

so that $M_{(\Lambda,h)} = \mathbb{C} v_{(\Lambda,h)}$. It is clear that these weight spaces are mutually orthogonal with respect to the contravariant form $B(.\,,\,.)$.

Denote by $\natural : \alpha \mapsto \alpha^\natural$ the projection of $\mathfrak{h}$ on $\mathfrak{h}^\natural$ with respect to decomposition (5.3). Then we have for $\alpha \in \mathfrak{h}$:

$$\alpha = (\alpha|x)\theta + \alpha^\natural. \tag{7.2}$$

Let $\Delta^\natural \subset \mathfrak{h}^{\natural *}$ be the set of roots of $\mathfrak{g}^\natural$ with respect to the Cartan subalgebra $\mathfrak{h}^\natural$ and let $\Delta_+^\natural$ be the subset of positive roots compatible with that of $\mathfrak{g}$. It follows from (5.10) that the "rho" for $\Delta_+^\natural$, i.e. the half of the difference between the sums of the sets of even and odd roots from $\Delta_+^\natural$, coincides with $\rho^\natural$. Let $\Delta' \subset \mathfrak{h}^{\natural *}$ be the set of weights of $\mathfrak{g}^\natural$ in $\mathfrak{g}_{1/2}$.

We define the set of roots Δ_W of the vertex algebra $W_k(\mathfrak{g}, x, f)$ as the disjoint union of three sets of pairs $(\alpha, m) \in \mathfrak{h}^{\natural *} \times \frac{1}{2}\mathbb{Z}$:

$$\Delta_W^\natural = \{(\alpha, m) \mid \alpha \in \Delta^\natural, m \in \mathbb{Z}\}, \quad \Delta'_W = \left\{(\alpha, m) \mid \alpha \in \Delta', m \in \tfrac{1}{2} + \mathbb{Z}\right\},$$
$$\Delta_W^{im} = \{(0, m) \mid m \in \mathbb{Z}\},$$

the multiplicity of a root being 1 for the first two sets (except for the case $\mathfrak{g} = sl(2|2)/\mathbb{C}I$ discussed in Section 8.4) and being $r = \dim \mathfrak{h}^\natural + 1$ for the third set. Define the subset of real positive roots Δ_W^{+re} (resp. all positive roots Δ_W^+) as the disjoint union of the first two (resp. all three) of the following subsets:

$$\Delta_W^{\natural +} = \{(\alpha, m) \mid \alpha \in \Delta^\natural, m \in \mathbb{N}\} \cup \{(\alpha, 0) \mid \alpha \in \Delta_+^\natural\},$$

$$\Delta_W'^{+} = \left\{(\alpha, m) \mid \alpha \in \Delta', m \in \frac{1}{2} + \mathbb{Z}_+\right\}, \quad \Delta_W^{im+} = \{(0, m) \mid m \in \mathbb{N}\}.$$

A root (α, m) is called odd if the corresponding to α root or weight vector is odd. We define the corresponding partition function $P_W(\eta)$, $\eta \in \mathfrak{h}^{\natural *} \times \frac{1}{2}\mathbb{Z}$, as the number of ways η can be represented in the form (counting multiplicities of roots):

$$\sum_{\alpha \in \Delta_W^+} k_\alpha \alpha, \quad \text{where } k_\alpha \in \mathbb{Z}_+, \text{ and } k_\alpha \leqslant 1 \text{ if } \alpha \text{ is odd.}$$

We denote by $\det_{(\eta,s)}(k, h, \Lambda)$ the determinant of the contravariant form $B(.\,,\,.)$ restricted to the weight subspace $M_{(\Lambda-\eta, h+s)}$, $\eta \in \mathbb{Z}_+\Delta_W^+$, $s \in \frac{1}{2}\mathbb{Z}_+$, of the Verma module M with highest weight (Λ, h). This is a function of the level k (or, in view of (5.7), of the central charge c), the lowest eigenvalue h of L_0 and of $\Lambda \in \mathfrak{h}^{\natural *}$.

In the determinant formula given below we use the normalization of the bilinear form $(.\,|\,.)$ on $\mathfrak{g}$ such that $(\theta|\theta) = 2$. The corresponding value of $h^\vee$ is given in Tables 1–3. The form $(.\,|\,.)$ restricted to $\mathfrak{h}^\natural$ is non-degenerate, and we identify $\mathfrak{h}^\natural$ with $\mathfrak{h}^{\natural *}$ using this form.

Theorem 7.2. *Let $\hat{\eta} \in \mathfrak{h}^{\natural *} \times \frac{1}{2}\mathbb{Z}$. Then, up to a non-zero constant factor, depending only on the choice of a basis of the weight space $M_{(\Lambda,h)-\hat{\eta}}$, the determinant $\det_{\hat{\eta}}(k, h, \Lambda)$ is given by the following formula*:

$$(k + h^\vee)^{(r-1)\sum_{m,n\in\mathbb{N}} P_W(\hat{\eta}-(0,mn))} \prod_{m,n\in\mathbb{N}} (h - h_{n,m}(k,\Lambda))^{P_W(\hat{\eta}-(0,mn))}$$
$$\times \prod_{m,n\in\mathbb{N}} \prod_{\beta\in\Delta_+^\natural} \varphi_{n,m,-\beta}(k,\Lambda)^{(-1)^{p(\beta)(n+1)} P_W(\hat{\eta}-n(-\beta,m))} \varphi_{n,m-1,\beta}(k,\Lambda)^{(-1)^{p(\beta)(n+1)} P_W(\hat{\eta}-n(\beta,m-1))}$$
$$\times \prod_{m\in\frac{1}{2}+\mathbb{Z}_+} \prod_{n\in\mathbb{N}} \prod_{\gamma\in\Delta'} (h - h_{n,m,\gamma}(k,\Lambda))^{(-1)^{p(\gamma)(n+1)} P_W(\hat{\eta}-n(\gamma,m))},$$

where

$$\varphi_{n,m,\beta}(k,\Lambda) = (\Lambda + \rho^\natural|\beta) + m(k + h^\vee) - \frac{n}{2}(\beta|\beta),$$

$$h_{n,m,\gamma}(k,\Lambda) = \frac{1}{4(k+h^\vee)}((2(\Lambda + \rho^\natural|\gamma) + 2m(k+h^\vee) - n(\gamma|\gamma))^2$$
$$- (k+1)^2 + 2(\Lambda|\Lambda + 2\rho^\natural)),$$

$$h_{n,m}(k,\Lambda) = \frac{1}{4(k+h^\vee)}((m(k+h^\vee)-n)^2 - (k+1)^2 + 2(\Lambda|\Lambda+2\rho^\natural)).$$

Theorem 7.2 follows from Theorem 7.1b and some lemmas on Verma modules over $\hat{\mathfrak{g}}$ stated below. Recall that a weight vector v_μ of a $\hat{\mathfrak{g}}$-module (resp. the weight μ) is called *singular* if $v_\mu \neq 0$ and v_μ is annihilated by $\hat{\mathfrak{n}}_+ := \mathfrak{n}_+ + \sum_{n\geqslant 1} \mathfrak{g}t^n$. Recall that the set of roots of $\hat{\mathfrak{g}}$ is $\hat{\Delta} = \{\alpha + mK \mid \alpha \in \Delta \cup \{0\}, m \neq 0 \text{ if } \alpha \in 0\}$. A root $\alpha + mK$ of $\hat{\mathfrak{g}}$ is positive if either $\alpha \in \Delta_+$ and $m \in \mathbb{Z}_+$ or $\alpha \in -\Delta_+ \cup \{0\}$ and $m \in \mathbb{N}$, it is even iff α is even, and it has multiplicity 1 unless $\alpha = 0$, in which case the multiplicity is $r = \dim \mathfrak{h}$.

Lemma 7.1 (Kac and Kazhdan [KK], Kac [K2]). *Let P be a Verma $\hat{\mathfrak{g}}$-module with highest weight $\hat{\lambda}$. Let $\tilde{\alpha}$ be a positive root of $\hat{\mathfrak{g}}$ and let n be a positive integer. Suppose that $(\hat{\lambda}+\hat{\rho}|\tilde{\alpha}) = \frac{n}{2}(\tilde{\alpha}|\tilde{\alpha})$. Then $\hat{\lambda} - n\tilde{\alpha}$ is a singular weight of P in the following cases*:

(i) *$\tilde{\alpha} = \alpha + mK$ is an even real (i.e., α is an even root) positive root, such that $\frac{1}{2}\alpha$ is not an odd root,*
(ii) *$\tilde{\alpha} = \alpha + mK$ is an odd positive root such that 2α is an even root and n is odd,*
(iii) *$\tilde{\alpha} = \alpha + mK$ is an odd positive root such that 2α is not a root and $n = 1$.*

Remark 7.1. This lemma follows from the determinant formula for the contravariant form on a weight space with weight $\lambda - \eta$ of a Verma module with highest weight λ over any contragradient Lie superalgebra $\mathfrak{g}(A,\tau)$ associated to a symmetrizable matrix A and the anti-involution $\omega(e_i) = (f_i),\ \omega(f_i) = e_i,\ \omega(h_i) = h_i$ [K2]:

$$\det{}_\eta(\lambda) = \prod_{\alpha\in\Delta_+}\prod_{n\in\mathbb{N}}\left((\lambda+\rho|\alpha) - \frac{n}{2}(\alpha|\alpha)\right)^{(-1)^{p(\alpha)(n+1)}P(\eta-n\alpha)\dim\mathfrak{g}_\alpha},$$

where P is the partition function for $\mathfrak{g}(A,\tau)$. (Unfortunately there is a misprint [K2] in the exponent of this formula: $n+1$ is missing there.) The determinant formula for $\hat{\mathfrak{g}}$, which is a special case of the above formula, is as follows:

$$\det{}_{\hat{\eta}}(\hat{\lambda}) = (k+h^\vee)^{\sum_{m,n\in\mathbb{N}} P(\hat{\eta}-mnK)} \prod_{n\in\mathbb{N}}\prod_{\tilde{\alpha}}((\hat{\lambda}+\hat{\rho}|\tilde{\alpha}) - \frac{n}{2}(\tilde{\alpha}|\tilde{\alpha}))^{P(\hat{\eta}-n\tilde{\alpha})}$$
$$\times \prod_{n\in 1+2\mathbb{Z}_+}\prod_{\tilde{\beta}}\left((\hat{\lambda}+\hat{\rho}|\tilde{\beta}) - \frac{n}{2}(\tilde{\beta}|\tilde{\beta})\right)^{P(\hat{\eta}-n\tilde{\beta})}\prod_{\tilde{\gamma}}(\hat{\lambda}+\hat{\rho}|\tilde{\gamma})^{P_{\tilde{\gamma}}(\hat{\eta}-\tilde{\gamma})},$$

where $\tilde{\alpha}$, $\tilde{\beta}$ and $\tilde{\gamma}$ run over the positive roots described in (i), (ii) and (iii), respectively, and $P_{\tilde{\gamma}}$ denotes the number of partitions not involving $\tilde{\gamma}$. Of course, both determinant formulas are given up to a non-zero factor depending on the choice of basis of the weight space. The latter determinant formula implies that for a generic $\hat{\lambda}$ on any hyperplane $k=a$, where $a\neq -h^\vee$, the Verma $\hat{\mathfrak{g}}$-module with highest weight $\hat{\lambda}$ is irreducible.

Lemma 7.2. *Let P be a Verma $\hat{\mathfrak{g}}$-module with highest weight vector $v_{\hat{\lambda}}$, and let $v_{\hat{\lambda}-n\tilde{\alpha}}$ be a singular vector corresponding to one of the singular weights described by Lemma* 7.1, *cases* (i)–(iii). *Then*

(a) *in cases* (i) *and* (ii), *the map $U(\hat{\mathfrak{n}}_-)\to P$ defined by $u\mapsto u(v_{\hat{\lambda}-n\tilde{\alpha}})$, is injective*,
(b) *in case* (iii), *the map $U^{-\tilde{\alpha}}(\hat{\mathfrak{n}}_-)\to P$, defined by $u\mapsto u(v_{\hat{\lambda}-\tilde{\alpha}})$, is injective, where $U^{-\tilde{\alpha}}(\hat{\mathfrak{n}}_-)$ denotes the span of all PBW monomials in negative root vectors, except for that corresponding to $-\tilde{\alpha}$.*

Proof. In all cases (i)–(iii) we have: $v_{\hat{\lambda}-n\tilde{\alpha}}=(e^n_{-\tilde{\alpha}}+\cdots)v_{\hat{\lambda}}$, where $e_{-\tilde{\alpha}}$ is a root vector attached to $-\tilde{\alpha}$ and $\cdots$ signify a linear combination of products of root vectors $e_{-\tilde{\beta}}$, where $0<\tilde{\beta}<\tilde{\alpha}$. $\square$

A weight vector of a $W_k(\mathfrak{g},e_{-\theta})$-module is called singular if it is annihilated by all "raising" operators L_n, $J_n^{\{a\}}$, $G_n^{\{v\}}$ for $n>0$ and $J_0^{\{a\}}$ for $a\in\mathfrak{n}_{0+}$ (and it is non-zero).

Lemma 7.3. *Let P be a Verma module over $\hat{\mathfrak{g}}$ with highest weight $\hat{\lambda}=\lambda+kD$ $(\lambda\in\mathfrak{g})$ of level k, and let $|0\rangle$ be the vacuum vector of $F(\mathfrak{g},x,f)$.*

(a) *Let u be a singular vector of the $\hat{\mathfrak{g}}$-module P with weight $\hat{\lambda}-n\tilde{\alpha}$, where $\tilde{\alpha}=\alpha+mK$ is as in Lemma* 7.1(i)–(iii) *and $(\alpha|x)=0,\ 1/2$ or 1. Then the homology class of the vector $u\otimes|0\rangle$ is a singular vector of the $W_k(\mathfrak{g},e_{-\theta})$-module $H(P)$.*
(b) *If u is a weight vector of P with weight $\hat{\mu}$, then the weight of the vector $u\otimes|0\rangle$ of the $W_k(\mathfrak{g},e_{-\theta})$-module $H(P)$ is*:

$$\left(\mu,\frac{(\hat{\lambda}|\hat{\lambda}+2\hat{\rho})}{2(k+h^\vee)}-\hat{\mu}(x+D)\right),\quad \textit{where } \mu=\hat{\mu}|_{\mathfrak{h}}.$$

(c) *The condition $(\hat{\lambda}+\hat{\rho}|mK+\alpha)=\frac{n}{2}(\alpha|\alpha)$, $\alpha\in\Delta$, $m,n\in\mathbb{Z}$, on $\hat{\lambda}$ is equivalent to the following condition on the highest weight (Λ,h) of the $W_k(\mathfrak{g},e_{-\theta})$-Verma module $H(P)$*:

$$\varphi_{n,m,\alpha^\natural}(k,\Lambda)=0 \quad \textit{if}\ \ (x|\alpha)=0,$$

$$h_{n,m+1/2,\alpha^\natural}(k,\Lambda)=h \quad \textit{if}\ \ (x|\alpha)=1/2,$$

$$h_{n,m+1}(k,\Lambda)=h \quad \textit{if}\ \ \alpha=\theta.$$

Proof. It is clear that if u is a singular vector of the $\hat{\mathfrak{g}}$-module P, then the vector $u \otimes |0\rangle$ of $\mathscr{C}(P)$ is d_0^P-closed. It is also clear that this vector is annihilated by all raising operators. It follows from the computation of $H(P)$ in Section 6 that if u is a singular vector of the type considered, the vector $u \otimes |0\rangle$ is not d_0^P-exact, proving (a).

Statement (b) follows from the following expression for L_0:

$$L_0 = \frac{\hat{\Omega}}{2(k+h^\vee)} - D - x + (\text{ghost terms}), \tag{7.3}$$

where $\hat{\Omega}$ is the Casimir operator for $\hat{\mathfrak{g}}$ [K3], hence has eigenvalue $(\hat{\lambda}|\hat{\lambda}+2\hat{\rho})$ on P.

Statement (c) is derived from (b) by a straightforward calculation. By (7.2) we have:

$$(\hat{\lambda}+\hat{\rho}|\hat{\lambda}+\hat{\rho}) = \tfrac{1}{2}(\lambda+\rho|\theta)^2 + |\lambda^\natural+\rho^\natural|^2. \tag{7.4}$$

Letting $\hat{\lambda}=0$ in (7.4), we get

$$(\hat{\rho}|\hat{\rho}) = \tfrac{1}{2}(h^\vee-1)^2 + (\rho^\natural|\rho^\natural). \tag{7.5}$$

Using (b), (7.4) and (7.5), we find h, the minimal eigenvalue of L_0:

$$h = \frac{1}{4(k+h^\vee)}((\lambda+\rho|\theta)-(k+h^\vee))^2 - (k+1)^2 + 2(\lambda^\natural|\lambda^\natural+2\rho^\natural). \tag{7.6}$$

Furthermore, we have by (7.2):

$$(\hat{\lambda}+\hat{\rho}|mK+\alpha) = m(k+h^\vee) + (\alpha|x)(\lambda+\rho|\theta) + (\lambda^\natural-\rho^\natural|\alpha^\natural). \tag{7.7}$$

In particular, we have, provided that $(\alpha|x)\neq 0$:

$$(\lambda+\rho|\theta) = \frac{1}{(\alpha|x)}((\hat{\lambda}+\hat{\rho}|mK+\alpha) - m(k+h^\vee) - (\lambda^\natural+\rho^\natural|\alpha^\natural)).$$

Plugging this in (7.6), we get, provided that $(\alpha|x)\neq 0$:

$$\begin{aligned} h = &\frac{1}{4(k+h^\vee)(\alpha|x)^2}((m+(\alpha|x))(k+h^\vee) \\ &- (\hat{\lambda}+\hat{\rho}|mK+\alpha) + (\lambda^\natural+\rho^\natural|\alpha^\natural))^2 \\ &- (\alpha|x)^2(k+1)^2 + \frac{(\lambda^\natural|\lambda^\natural+2\rho^\natural)}{2(k+h^\vee)}. \end{aligned} \tag{7.8}$$

By (7.7) and (7.8), the condition $(\hat{\lambda}+\hat{\rho}|mK+\alpha)=\frac{n}{2}(\alpha|\alpha)$ is equivalent to:

$$\varphi_{n,m,\alpha^{\natural}}(k,\lambda^{\natural})=0 \quad \text{if } (\alpha|x)=0,$$

$$h=h_{n,m+1}(k,\lambda^{\natural}) \quad \text{if } \alpha=\theta,$$

$$h=h_{n,m+1/2,\alpha^{\natural}}(k,\lambda^{\natural}) \quad \text{if } (\alpha|x)=1/2. \qquad \square$$

Proof of Theorem 7.2. The proof follows the traditional lines, cf. e.g. [KR]. Choose an ordered basis $\{u_\alpha \mid \alpha\in\Delta^{\natural}\}\coprod\{u_0^i \mid i=1,\dots,r-1\}$ of $\mathfrak{g}^{\natural}$ consisting of root vectors and a basis of $\mathfrak{h}^{\natural}$, such that $(\omega(u_\alpha)|u_\beta)=\delta_{\alpha\beta}$ and $(u_0^i \mid u_0^j)=\delta_{ij}$. Choose an ordered basis $\{v_\gamma \mid \gamma\in\Delta'\}$ of $\mathfrak{g}_{-1/2}$ consisting of weight vectors such that $\langle\omega(v_\beta),v_\gamma\rangle_{\mathrm{ne}}=\delta_{\beta\gamma}$.

Due to Theorem 5.1, the vertex algebra is strongly generated by the fields of the following three types:

$$L(z)=\sum_{m\in\mathbb{Z}}L_m z^{-m-2},$$
$$J^{\{u_\alpha\}}(z)=\sum_{m\in\mathbb{Z}}J_m^{\{u_\alpha\}}z^{-m-1}, \quad J^{\{u_0^i\}}=\sum_{m\in\mathbb{Z}}J_m^{\{u_0^i\}}z^{-m-1},$$
$$\frac{1}{(k+h^\vee)^{1/2}}G^{\{v_\gamma\}}=\sum_{m\in\frac{1}{2}+\mathbb{Z}}G_m^{\{v_\gamma\}}z^{-m-3/2}.$$

We shall need the following commutation relations:

$$[L_m,L_{-m}]=2mL_0+\frac{m^3-m}{12}c(k), \quad \text{where } c(k) \text{ is given by (5.7)}, \tag{7.9}$$

$$[J_m^{\{u\}},J_{-m}^{\{u'\}}]=J_0^{\{[u,u']\}}+m(k+\tfrac{1}{2}(h^\vee-\sum_i h_{0,i}^\vee))(u|u') \text{ (see Theorem 5.1d)}, \tag{7.10}$$

$$[G_m^{\{v\}},G_{-m}^{\{v'\}}]=-2(e|[v,v'])L_0+\frac{1}{k+h^\vee}A(v,v',m), \tag{7.11}$$

where $A(v,v',m)$ is a (possibly infinite) linear combination over $\mathbb{C}$ of monomials of the form $J_{-n}^{\{u_\alpha\}}J_n^{\{u_\beta\}}$ with $n>0$ plus a polynomial of degree at most 2 in $J_0^{\{u_\alpha\}}$ and k, not involving k^2 (see Theorem 5.1e).

Let M be a Verma module over $W_k(\mathfrak{g},e_{-\theta})$ with highest weight vector $v_{\Lambda,h}$. Recall that the following vectors form a basis of M compatible with its weight space decomposition (7.1):

$$(G_{-r_1}^{\{v_{\gamma_1}\}})^{c_1}(G_{-r_2}^{\{v_{\gamma_2}\}})^{c_2}\dots(J_{-n_1}^{\{u_{\alpha_1}\}})^{b_1}(J_{-n_2}^{\{u_{\alpha_2}\}})^{b_2}\dots(L_{-m_1})^{a_1}(L_{-m_2})^{a_2}\dots v_{(\Lambda,h)}, \tag{7.12}$$

where $a_i, b_i, c_i \in \mathbb{Z}_+$ and b_i (resp. c_i) $\leqslant 1$ if α_i (resp. γ_i) is odd; $m_i, n_i, r_i > 0$ or $n_i = 0$ and α_i is a negative root; and each sequence $m_1, m_2, \ldots$; (α_1, n_1), $(\alpha_2, n_2), \ldots$; (γ_1, r_1), $(\gamma_2, r_2), \ldots$ is strictly decreasing. Note that in the weight space decomposition (7.1) the vector (7.12) has weight

$$\left(\Lambda - \sum n_i\alpha_i - \sum r_i\gamma_i, h + \sum m_i + \sum n_i + \sum r_i\right). \tag{7.13}$$

The function $\det_{\hat{\eta}}(k, h, \Lambda)$ is a rational function in k and a polynomial function in h and Λ. First, we compute the total degree of this function (by letting the degree of k, h and Λ be 1 and defining the degree of a rational function as the difference between the degrees of the numerator and the denominator).

Let $\{R_i\}$ be the basis of the weight space $M_{(\Lambda,h)-\hat{\eta}}$ consisting of monomials (7.12) such that (7.13) is equal to $(\Lambda, h) - \hat{\eta}$. Then

$$\det_{\hat{\eta}}(k, h, \Lambda) = \det(B(R_i, R_j))_{i,j}.$$

It is clear from (7.9)–(7.11) that the leading term of $\det_{\hat{\eta}}$ comes from the diagonal of the matrix $(B(R_i, R_j))_{i,j}$, and that the leading term of each diagonal term $B(R_i, R_i)$ is the product of leading terms of factors of (7.12), i.e., the following three types of factors:

$$B(L_{-m}^n v_{(\Lambda,h)}, L_{-m}^n v_{(\Lambda,h)}), B(J_{-m}^{\{u_\alpha\}n} v_{(\Lambda,h)}, \quad J_{-m}^{\{u_\alpha\}n} v_{(\Lambda,h)}), \quad B(G_{-m}^{\{v_\gamma\}n} v_{(\Lambda,h)}, \quad G_{-m}^{\{v_\gamma\}n} v_{(\Lambda,h)}).$$

It is clear from (7.9)–(7.11) that each of these factors contributes n to the total degree of $\det_{\hat{\eta}}$. Hence the contribution to the total degree of all first type factors is equal to

$$\sum_{n\in\mathbb{N}} \sum_{m\in\mathbb{N}} n(P_W(\hat{\eta} - n(0,m)) - P_W(\hat{\eta} - (n+1)(0,m))) = \sum_{n\in\mathbb{N}} \sum_{m\in\mathbb{N}} P_W(\hat{\eta} - n(0,m)).$$

Likewise, the second (and resp. third) type factors with u (resp. v) even contribute to the total degree the number

$$\sum_{n\in\mathbb{N}} \sum_{\alpha\in\Delta^{\natural+}_{W,\mathrm{even}}} P_W(\hat{\eta} - n\alpha) \quad \left(\text{resp.} \sum_{\gamma\in\Delta'^{+}_{W,\mathrm{even}}} P_W(\hat{\eta} - n\gamma)\right).$$

If u (resp. v) is odd, then the second (resp. third) type factors contribute

$$\sum_{\alpha\in\Delta^{\natural+}_{W,\mathrm{odd}}} P_{W,\alpha}(\hat{\eta} - \alpha) \quad \left(\text{resp.} \sum_{\gamma\in\Delta'^{+}_{W,\mathrm{odd}}} P_{W,\gamma}(\hat{\eta} - \gamma)\right),$$

where $P_{W,\alpha}$ denotes the number of partitions by elements from $\Delta_W^+\backslash\{\alpha\}$. But if α is an odd root, we have:

$$P_W(\hat{\eta}) = P_{W,\alpha}(\hat{\eta}) + P_{W,\alpha}(\hat{\eta} - \alpha),$$

therefore

$$P_{W,\alpha}(\hat{\eta}-\alpha)=\sum_{n\in\mathbb{N}}(-1)^{n+1}P_W(\hat{\eta}-n\alpha).$$

Thus, the total degree of $\det_{\hat{\eta}}$ is equal to

$$T=\sum_{\alpha\in\Delta_W^+}\sum_{n\in\mathbb{N}}(-1)^{p(\alpha)(n+1)}P_W(\hat{\eta}-n\alpha). \tag{7.14}$$

Of course, in (7.14) and the previous formulas the α in the summations are counted with their multiplicities, namely $\alpha\in\Delta_W^{im+}$ is counted r times.

The same arguments show that the total degree in h and Λ of the leading term of $\det_{\hat{\eta}}$ is equal to

$$T'=T-(r-1)\sum_{m,n\in\mathbb{N}}P_W(\hat{\eta}-(0,mn)).$$

But Lemmas 7.1, 7.2 and 7.3 give us a factor of $\det_{\hat{\eta}}$ whose leading term has total degree in h and Λ at least T' (since singular vectors that are not highest weight vectors lie in the maximal submodule of $H(P)$). (The corresponding to $m\delta+\alpha$ root is $(\alpha^{\natural},m+\alpha(x))$.) It follows that, up to a factor, which is a rational function in k with poles only at $k=-h^{\vee}$, $\det_{\hat{\eta}}$ is given by Theorem 7.2.

In order to compute this factor, note that the free field realization given by Theorem 5.2 is irreducible for a generic Verma $V_{\alpha_k}(\mathfrak{g}_0)$-module, provided that for each simple component of $\mathfrak{g}_0$ (including the commutative ones) the level plus the dual Coxeter number $h_0^{\vee}$ of this component is non-zero. But for a simple component its level is $k+h^{\vee}-h_0^{\vee}$, hence $(k+h^{\vee}-h_0^{\vee})+h_0^{\vee}=k+h^{\vee}$ for all components. Thus, on any hyperplane $k+h^{\vee}=a$, $a\neq 0$, the generic Verma $V_{\alpha_k}(\mathfrak{g}_0)$-module is irreducible. Hence the factor may vanish only if $k+h^{\vee}=0$, hence the factor is a power of $k+h^{\vee}$. $\square$

Remark 7.2. In the case when $W(\mathfrak{g},e_{-\theta})$ is from Table 2, i.e. $\mathfrak{g}_{1/2}$ is purely odd, the determinant formula, given by Theorem 7.2 can be a bit simplified, using the fact that all elements from $\Delta_{-1/2}$ different from $-\theta/2$ are isotropic. Let $\varepsilon=2$ if $0\in\Delta'$ and $\varepsilon=1$ otherwise. Then the determinant $\det_{\hat{\eta}}(k,h,\Lambda)$ is given by the following formula:

$$(k+h^{\vee})^{(r-1)\sum_{m,n\in\mathbb{N}}P_W(\hat{\eta}-(0,mn))}\prod_{\substack{m,n\in\varepsilon^{-1}\mathbb{N}\\ m-n\in\mathbb{Z}}}(h-h_{n,\varepsilon m}(k,\Lambda))^{P_W(\hat{\eta}-(0,\varepsilon mn))}$$
$$\times\prod_{m,n\in\mathbb{N}}\prod_{\beta\in\Delta_+^{\natural}}\varphi_{n,m,-\beta}(k,\Lambda)^{P_W(\hat{\eta}-n(-\beta,m))}\varphi_{n,m-1,\beta}(k,\Lambda)^{P_W(\hat{\eta}-n(\beta,m-1))}$$
$$\times\prod_{m\in\frac{1}{2}+\mathbb{Z}_+}\prod_{\gamma\in\Delta'\setminus\{0\}}(h-h_{m,\gamma}(k,\Lambda))^{P_{W,(\gamma,m)}(\hat{\eta}-(\gamma,m))},$$

where

$$\varphi_{n,m,\beta}(k,\Lambda) = (\Lambda + \rho^{\natural}|\beta) + m(k+h^{\vee}) - \frac{n}{2}(\beta|\beta),$$

$$h_{m,\gamma}(k,\Lambda) = \frac{1}{4(k+h^{\vee})}((2(\Lambda+\rho^{\natural}|\gamma) + 2m(k+h^{\vee}))^2 - (k+1)^2 + 2(\Lambda|\Lambda+2\rho^{\natural})),$$

$$h_{n,m}(k,\Lambda) = \frac{1}{4(k+h^{\vee})}((m(k+h^{\vee})-n)^2 - (k+1)^2 + 2(\Lambda|\Lambda+2\rho^{\natural})).$$

8. Examples

8.1. Virasoro algebra

Recall that the Virasoro algebra is $W_k(s\ell_2, e_{-\theta})$. In this case $\dim \mathfrak{g}_0 = 1$, $\mathfrak{g}_{1/2} = 0$ and $h^{\vee} = 2$. We let $\mathfrak{g}_0 = \mathbb{C}a$, where $(a|a) = 1$. Then $x = a/\sqrt{2}$ since $(x|x) = 1/2$. The Virasoro central charge is given by (5.7):

$$c = 1 - \frac{6(k+1)^2}{k+2}.$$

The free field realization is given by Theorem 5.2:

$$L = \frac{1}{2(k+2)} : aa : + \frac{k+1}{\sqrt{2}(k+2)}\partial a,$$

where a is a free boson: $[a_{\lambda}a] = \lambda(k+2)$. We can remove the singularity at $k=-2$ by letting $b = \frac{1}{(k+2)^{1/2}}a$, so that $[b_{\lambda}b] = \lambda$, and

$$L = \frac{1}{2} : bb : + \left(\frac{1-c}{12}\right)^{1/2}\partial b.$$

This is the free field realization of the Virasoro algebra, which goes back to Virasoro and Fairlie.

The determinant formula given by Remark 7.2 looks as follows:

$$\det{}_N(k,h) = \prod_{m,n\in\mathbb{N}} (h - h_{n,m}(k))^{p(N-mn)}, \quad N \in \mathbb{Z}_+,$$

where p is the classical partition function and

$$h_{n,m}(k) = \frac{1}{4(k+2)}((m(k+2)-n)^2 - (k+1)^2).$$

This is the Kac determinant formula [K2,KR]. Note that it was observed already in [FF2] that under the quantum reduction the Kac–Kazhdan equations for $\hat{sl}_2$ are transformed to Kac equations for the Virasoro algebra.

8.2. Neveu–Schwarz algebra

Recall that the Neveu–Schwarz algebra is $W_k(spo(2|1), e_{-\theta})$. In this case $\mathfrak{g}_0$ is the 1-dimensional Lie algebra, $\mathfrak{g}_{1/2}$ is 1-dimensional and purely odd, and $h^\vee = 3/2$. We use the notation of [KRW], Section 6, except for a different normalization of the invariant bilinear form, namely $(a|b) = \operatorname{str} ab$. Then for $h = \theta$ we have $(h|h) = 2$, and we have a free neutral fermion Φ such that $[\Phi_\lambda \Phi] = 1$. The vertex algebra $W_k(spo(2|1), e_{-\theta})$ is strongly generated by two fields:

$$J^{\{f_\alpha\}} = J^{(f_\alpha)} + \tfrac{1}{2} : \Phi J^{(h)} : + (k+1)\partial\Phi,$$

$$J^{\{f_{2\alpha}\}} = J^{(f_{2\alpha})} + : \Phi J^{(f_\alpha)} : - \frac{1}{4} : J^{(h)} J^{(h)} : - \frac{k+1}{2} \partial J^{(h)} + \frac{2k+3}{4} : \Phi\partial\Phi : .$$

The fields $L = -\frac{1}{k+3/2} J^{\{f_{2\alpha}\}}$ and $G = \frac{1}{(k+3/2)^{1/2}} J^{\{f_\alpha\}}$ satisfy the relations of the Neveu–Schwarz algebra:

$$[L_\lambda L] = (\partial + 2\lambda)L + \frac{\lambda^3}{12} c, \quad [L_\lambda G] = \left(\partial + \frac{3}{2}\lambda\right) G, \quad [G_\lambda G] = L + \frac{\lambda^2}{6} c,$$

where the central charge is (see (5.7)):

$$c = \frac{3}{2} - \frac{12(k+1)^2}{2k+3}.$$

As in Theorem 6.2, we project to the tensor product of the affine vertex algebra associated to $\mathfrak{g}_0$ and $F^{\mathrm{ne}}(\mathfrak{g}_{1/2})$ to obtain the free field realization. In order to remove the singularity at $k = -3/2$, we let $b = J^{(h)}/(2k+3)^{1/2}$, so that $[b_\lambda b] = \lambda$, and let $\gamma = (k+1)/(2k+3)^{1/2}$. As before, $[\Phi_\lambda \Phi] = 1$. Then we get:

$$L = \tfrac{1}{2} : bb : + \gamma\partial b - \tfrac{1}{2} : \Phi\Phi :, G = \tfrac{1}{\sqrt{2}} : b\Phi : + \sqrt{2}\gamma\partial\Phi,$$

the central charge being $c = 3/2 - 12\gamma^2$. This is the free field realization of the Neveu–Schwarz algebra which goes back to Neveu–Schwarz [NS] and Thorn [T] (cf. [K4]).

The determinant formula given by Remark 7.2 looks as follows (cf. [K2,KW0]):

$$\det{}_N(k,h) = \prod_{\substack{m,n \in \frac{1}{2}\mathbb{N} \\ m-n \in \mathbb{Z}}} (h - h^{NS}_{n,m}(k))^{\tilde{p}(N-2mn)}, \quad N \in \frac{1}{2}\mathbb{Z}_+,$$

where $\tilde{p}(s)$ denotes the number of partitions of $s \in \frac{1}{2}\mathbb{Z}_+$ in a sum of parts from $\frac{1}{2}\mathbb{N}$, the non-integer parts being distinct, and

$$h^{NS}_{n,m}(k) = \frac{1}{2(2k+3)}((m(2k+3)-n)^2 - (k+1)^2).$$

8.3. $N = 2$ superconformal algebra

Recall that the $N = 2$ superconformal algebra is $W_k(s\ell(2|1), e_{-\theta})$ [KRW]. In this case $\mathfrak{g}_0$ is the 2-dimensional commutative Lie algebra, $\mathfrak{g}_{1/2}$ is 2-dimensional and purely odd, and $h^\vee = 1$. The Virasoro central charge is given by (5.7):

$$c = -3(2k+1).$$

Recall that the simple root vectors of $\mathfrak{g}$ are h_1, h_2 with scalar products $(h_1|h_1) = (h_2|h_2) = 0,\ (h_1|h_2) = 1$. We have: $\mathfrak{h} = \mathbb{C}h_1 + \mathbb{C}h_2,\ S_{1/2} = \{h_1, h_2\},\ S_0 = \emptyset,\ \theta = h_1 + h_2$. Let $\alpha = \frac{1}{2}(h_1 - h_2)$. Then $\mathfrak{h}^\natural = \mathbb{C}\alpha,\ \rho^\natural = 0,\ h_1^\natural = -h_2^\natural = \alpha$. We have two free neutral fermions $\Phi_1,\ \Phi_2$ with the λ-brackets:

$$[\Phi_{1\lambda}\Phi_2] = -1, \quad [\Phi_{i\lambda}\Phi_i] = 0.$$

The construction of the vertex algebra in question given by Theorem 5.1, is explicitly written down in [KRW], Section 7 (see (7.1) and formulas preceding it). To obtain the free field realization we project on the affine vertex algebra associated to $\mathfrak{g}_0$ to get $\gamma = (-k-1)^{1/2}$:

$$J = (h_1 - h_2) + :\Phi_1\Phi_2:, \quad G^+ = -\frac{1}{\gamma} :\Phi_2 h_1: + \gamma\partial\Phi_2, \quad G^- = -\frac{1}{\gamma} :\Phi_1 h_2: + \gamma\partial\Phi_1,$$

$$L = -\frac{1}{\gamma^2} :h_1h_2: + \frac{1}{2}\partial(h_1 + h_2) - \frac{1}{2}(:\Phi_1\partial\Phi_2: + :\Phi_2\partial\Phi_1:).$$

Note that the cubic terms in $G^\pm$ disappear since $\dim \mathfrak{g}_{1/2} < 3$. The filed L is a Virasoro filed with central charge c, the fields J (resp. $G^\pm$) are primary of conformal weight 1 (resp. 3/2), and the remaining λ-brackets are as follows:

$$[J_\lambda J] = \frac{\lambda}{3}c, [G^\pm{}_\lambda G^\pm] = 0, [J_\lambda G^\pm] = \pm G^\pm, [G^+{}_\lambda G^-] = L + \left(\frac{1}{2}\partial + \lambda\right)J + \frac{\lambda^2}{6}c,$$

which is the $N = 2$ superconformal algebra. We have: $\omega(J_n) = J_{-n}$, $\omega(G_n^\pm) = G_{-n}^\mp$.

In order to make the formulas more symmetric and to remove the singularity at $k = -1$, we let:

$$b^+ = -\frac{1}{\gamma}h_1, \quad b^- = \frac{1}{\gamma}h_2, \quad \psi^+ = \Phi_2, \quad \psi^- = \Phi_1,$$

so that $[b^{\pm}{}_{\lambda}b^{\mp}] = \lambda$, $[b^{\pm}{}_{\lambda}b^{\pm}] = 0$, $[\psi^{\pm}{}_{\lambda}\psi^{\mp}] = 1$, $\psi^{\pm}{}_{\lambda}\psi^{\pm} = 0$. Then we get the free field realization by Kato–Matsuda [KM] (cf. [K4]):

$$J = -\gamma(b^+ + b^-) + :\psi^+\psi^-:, \quad G^+ =: \psi^+ b^+ : +\gamma\partial\psi^+, \quad G^- =: \psi^- b^- : -\gamma\partial\psi^-,$$

$$L =: b^+ b^- : +\frac{\gamma}{2}\partial(b^- - b^+) + \frac{1}{2}(:\psi^-\partial\psi^+ : + : \psi^+\partial\psi^- :),$$

the central charge being $c = 3 + 6\gamma^2$.

Using the non-Dynkin gradation of $s\ell(2|1)$, discussed in [KRW] as well, we get another "free field" realization of the $N = 2$ superconformal algebra. In this case $\mathfrak{g}_0 \simeq g\ell(1|1)$, where $\mathbb{C}^{1|1} = \mathbb{C}\varepsilon_1 + \mathbb{C}\varepsilon_2$, ε_1 even, ε_2 odd, we take the bilinear form $(a|b) = \mathrm{str}\, ab$ on $\mathfrak{g}_0$, and the following basis:

$$e = E_{12}, \quad f = E_{21}, \quad h_1 = E_{11} + E_{22}, \quad h_2 = E_{11}.$$

Consider the associated affine Lie superalgebra $\hat{\mathfrak{g}}_0$ of level k. Then we have the following "free field" realization in $\hat{\mathfrak{g}}_0$ with $c = -6k + 3$ (cf. [KRW]):

$$J = h_1 - h_2, \quad G^+ = -\frac{1}{k}f, \quad G^- =: h_2 e : -(k-1)\partial e,$$

$$L = -\frac{1}{k}(: ef : - : h_1 h_2 :) + \frac{1}{2}\partial(h_1 + h_2).$$

In order to write down the determinant formula, let $\Lambda = -j\alpha$, where $j \in \mathbb{C}$, so that

$$(\Lambda|\Lambda) = -j^2/2, \quad (\Lambda|h_1 - h_2) = j, \quad (\Lambda|\alpha) = j/2.$$

The set of positive roots Δ_W^+ consists of the set of even positive roots $\{(0,m)\,|\, m \in \mathbb{N}\}$ and the set of odd positive roots $\{(\pm\alpha, m)\,|\, m \in \frac{1}{2} + \mathbb{Z}_+\}$, all of multiplicity 1. The partition function $P_W(\hat{\eta})$ is equal to the number of partitions of $\hat{\eta} = (m\alpha, n)$ in a sum of roots from Δ_W^+, the odd roots being distinct.

The highest weight (Λ, h) of a Verma module over $N = 2$ superconformal algebra is determined by $h =$ the lowest eigenvalue of L_0 and $j =$ the eigenvalue of J_0^0 on the highest weight vector.

The factors that occur in Remark 7.2 are $k+1$, $h - h_{n,m}(k,j)$ and $h - h_{n,\pm\alpha}(k,j)$, where

$$h_{n,m}(k,j) = \frac{1}{4(k+1)}((m(k+1) - n)^2 - (k+1)^2 - j^2),$$

$$h_{m,\pm\alpha}(k,j) = \frac{1}{4(k+1)}((\pm j + 2m(k+1))^2 - (k+1)^2 - j^2)$$

$$= \left(m^2 - \frac{1}{4}\right)(k+1) \pm jm.$$

Hence Remark 7.2 gives the following determinant formula (cf. [KM]):

$$\det_{\hat{\eta}}(k,h,j) = \prod_{m,n\in\mathbb{N}} \left(4(k+1)h - (m(k+1)-n)^2 + (k+1)^2 + j^2\right)^{P_W(\hat{\eta}-(0,mn))}$$
$$\times \prod_{m\in\frac{1}{2}+\mathbb{Z}_+} \left(h - \left(\left(m^2 - \frac{1}{4}\right)(k+1) + jm\right)\right)^{P_{W,(\alpha,m)}(\hat{\eta}-(\alpha,m))}$$
$$\times \left(h - \left(\left(m^2 - \frac{1}{4}\right)(k+1) - jm\right)\right)^{P_{W,(-\alpha,m)}(\hat{\eta}-(-\alpha,m))}.$$

8.4. $N = 4$ *superconformal algebra*

As we shall see, the $N=4$ superconformal algebra is $W_k(\mathfrak{g}, e_{-\theta})$, where $\mathfrak{g} = s\ell(2|2)/\mathbb{C}I$. In this case $\mathfrak{g}^{\natural} = s\ell_2$, $\mathfrak{g}_{1/2}$ is 4-dimensional purely odd, and as an $s\ell_2$-module it is a direct sum of two 2-dimensional irreducible $s\ell_2$-modules. The dual Coxeter number $h^\vee = 0$, hence by (5.7), the Virasoro central charge is:

$$c = -6(k+1).$$

The simple roots of $\mathfrak{g}$ are $\alpha_1,\ \alpha_2,\ \alpha_3$, where α_1 and α_3 are odd and α_2 is even, and the non-zero scalar products between them are as follows:

$$(\alpha_1|\alpha_2) = 1, \quad (\alpha_2|\alpha_3) = 1, \quad (\alpha_2|\alpha_2) = -2.$$

The subalgebra $\mathfrak{g}^{\natural} = s\ell_2$ corresponds to the root α_2, so that its dual Coxeter number $h_0^\vee = -2$, and $\mathfrak{h}^{\natural} = \mathbb{C}\alpha_2$.

We choose root vectors e_α and $e_{-\alpha}$ ($\alpha \in \Delta_+$) such that $(e_\alpha|e_{-\alpha}) = -1$, so that $[e_{\alpha\lambda} e_{-\alpha}] = -\alpha - \lambda K$. We use the following notations: $e_i = e_{\alpha_i}$, $f_i = e_{-\alpha_i}$, $e_{ij} = e_{\alpha_i+\alpha_j}$, $f_{ij} = e_{-\alpha_i-\alpha_j}$, etc., $h_i = -\alpha_i$, $h_{ij} = -\alpha_i - \alpha_j$, etc. We have: $\theta = \alpha_1 + \alpha_2 + \alpha_3$, so that $x = -\frac{1}{2} h_{123}$, and we take $f = f_{123}$. We have 4 free neutral fermions:

$$\Phi_1 = \Phi_{\alpha_1}, \Phi_3 = \Phi_{\alpha_3}, \quad \Phi_{12} = \Phi_{\alpha_1+\alpha_2}, \quad \Phi_{23} = \Phi_{\alpha_2+\alpha_3},$$

whose λ-brackets are:

$$[\Phi_{\alpha\lambda}\Phi_\beta] = 1 \quad \text{if } \alpha+\beta = \theta, \text{ and } = 0 \text{ otherwise},$$

so that $\Phi^\alpha = \Phi_{\theta-\alpha}$.

Then we have the following d_0-closed fields, which strongly generate $W_k(\mathfrak{g}, e_{-\theta})$ (by Theorem 4.1):

$$J^0 = J^{(h_2)} - :\Phi_1\Phi_{23}: - :\Phi_3\Phi_{12}:, \quad J^+ = J^{(e_2)} - :\Phi_{12}\Phi_{23}:, \quad J^- = J^{(f_2)} - :\Phi_1\Phi_3:,$$

$$G^+ = \frac{1}{\sqrt{k}}(J^{(f_1)} + :\Phi_{23}J^{(h_1)}: - :\Phi_3 J^{(e_2)}: - (k+1)\partial\Phi_{23} + :\Phi_3\Phi_{12}\Phi_{23}:),$$

$$G^- = \frac{1}{\sqrt{k}}(J^{(f_{12})} - :\Phi_{23}J^{(f_2)}: + :\Phi_3 J^{(h_{12})}: - (k+1)\partial\Phi_3 + :\Phi_1\Phi_3\Phi_{23}:),$$

$$\bar{G}^+ = \frac{1}{\sqrt{k}}(J^{(f_3)} + :\Phi_{12}J^{(h_3)}: + :\Phi_1 J^{(e_2)}: - (k+1)\partial\Phi_{12} - :\Phi_1\Phi_{12}\Phi_{23}:),$$

$$\bar{G}^- = \frac{1}{\sqrt{k}}(J^{(f_{23})} + :\Phi_{12}J^{(f_2)}: + :\Phi_1 J^{(h_{23})}: - (k+1)\partial\Phi_1 - :\Phi_1\Phi_3\Phi_{12}:),$$

$$L = -\frac{1}{k}\left(J^{(f)} + \sum_{\alpha\in S_{1/2}} :\Phi_\alpha J^{(f_\alpha)}:\right) + \frac{1}{4k}(:J^{(h_{123})}J^{(h_{123})}: - :J^{(h_2)}J^{(h_2)}:)$$
$$- \frac{1}{2k}(:J^{(e_2)}J^{(f_2)}: + :J^{(f_2)}J^{(e_2)}:) - \frac{k+1}{2k}\partial J^{(h_{123})} + \frac{1}{2}\sum_{\alpha\in S_{1/2}} :\Phi_\alpha\partial\Phi_{\theta-\alpha}: .$$

The field L is a Virasoro field with central charge c, the fields J^0, $J^{\pm}$ (resp. $G^{\pm}$, $\bar{G}^{\pm}$) being primary of conformal weight 1 (resp. $3/2$). The remaining non-zero λ-brackets are as follows:

$$[J^0{}_\lambda J^{\pm}] = \pm 2J^{\pm}, \quad [J^0{}_\lambda J^0] = \frac{\lambda}{3}c, \quad [J^+{}_\lambda J^-] = J^0 + \frac{\lambda}{6}c,$$

$$[J^0{}_\lambda G^{\pm}] = \pm G^{\pm}, \quad [J^0{}_\lambda \bar{G}^{\pm}] = \pm \bar{G}^{\pm},$$

$$[J^+{}_\lambda G^-] = G^+, \quad [J^-{}_\lambda G^+] = G^-, \quad [J^+{}_\lambda \bar{G}^-] = -\bar{G}^+, \quad [J^-{}_\lambda \bar{G}^+] = -\bar{G}^-,$$

$$[G^{\pm}{}_\lambda \bar{G}^{\pm}] = (\partial + 2\lambda)J^{\pm}, \quad [G^{\pm}{}_\lambda \bar{G}^{\mp}] = L \pm \frac{1}{2}(\partial + 2\lambda)J^0 + \frac{\lambda^2}{6}c.$$

These are the λ-brackets of the $N=4$ superconformal algebra (cf. [K4]). We have: $\omega(J^0_n) = J^0_{-n}$, $\omega(J^{\pm}_n) = J^{\mp}_{-n}$, $\omega(G^{\pm}_n) = \bar{G}^{\mp}_{-n}$.

By Theorem 5.2, the free field realization is given in the vertex algebra

$$V_{-k-2}(s\ell_2) \otimes B \otimes F^{\mathrm{ne}},$$

where $V_{-k-2}(s\ell_2)$ is the universal affine vertex algebra associated to $s\ell_2$, with the standard basis E, H, F and the standard bilinear form $(a|b) = \mathrm{tr}\, ab$, of level $-k-2$ $(= -(k + h^\vee - h_0^\vee))$, B is the vertex algebra strongly generated by the free boson $\theta(z)$ and F^{ne} is the vertex algebra strongly generated by the free fermions Φ_1, Φ_3, Φ_{12}, Φ_{23}. Explicit formulas are obtained from the above formulas for the d_0-closed fields by projecting on this vertex algebra:

$$J^0 = H - :\Phi_1\Phi_{23}: - :\Phi_3\Phi_{12}:, \quad J^+ = E - :\Phi_{12}\Phi_{23}:, \quad J^- = F - :\Phi_1\Phi_3:,$$

$$G^+ = \frac{1}{\sqrt{k}}\left(\frac{1}{2}:\Phi_{23}(\theta - H): - :\Phi_3 E: - (k+1)\partial\Phi_{23} + :\Phi_3\Phi_{12}\Phi_{23}:\right),$$

$$G^- = \frac{1}{\sqrt{k}}\left(\frac{1}{2} : \Phi_3(\theta + H) : - : \Phi_{23}F : -(k+1)\partial\Phi_3 + : \Phi_1\Phi_3\Phi_{23}\right) :,$$

$$\bar{G}^+ = \frac{1}{\sqrt{k}}\left(\frac{1}{2} : \Phi_{12}(\theta - H) : + : \Phi_1 E : -(k+1)\partial\Phi_{12} - : \Phi_1\Phi_{12}\Phi_{23} :\right),$$

$$\bar{G}^- = \frac{1}{\sqrt{k}}\left(\frac{1}{2} : \Phi_1(\theta + H) : + : \Phi_{12}F : -(k+1)\partial\Phi_1 - : \Phi_1\Phi_3\Phi_{12} :\right),$$

$$L = \frac{1}{4k}(: \theta\theta : - : HH : -2 : EF : -2 : FE :) - \frac{k+1}{2k}\partial\theta + \frac{1}{2}(: \Phi_1\partial\Phi_{23} : + : \Phi_3\partial\Phi_{12} : + : \Phi_{12}\partial\Phi_3 : + : \Phi_{23}\partial\Phi_1 :).$$

Comparing with the character formulas of [ET], we see that, taking a unitary representation of $\hat{s\ell_2}$ and that of B such that $l < h$, where l is the isospin and h is the lowest eigenvalue of L_0, the above formulas give an explicit construction of all massive unitary representations of the $N = 4$ superconformal algebra. In the case of equality, i.e., the massless unitary representations, this construction is finitely reducible.

Of course, the genuine free field realization is obtained by making use of the free field realization of $\hat{s\ell_2}$ given in [W].

Let $\alpha = \alpha_2|_{\mathfrak{h}^\natural}$. Then $\Delta^\natural = \{\pm\alpha\}$, where $\pm\alpha$ are even of multiplicity 1, and $\Delta' = \{\pm\alpha/2\}$, where $\pm\alpha/2$ are odd of multiplicity 2. The set of positive roots Δ_W^+ consists of the set of even roots

$$\{(\pm\alpha, m) \mid m \in \mathbb{N}\} \cup \{(\alpha, 0)\} \cup \{(0, m) \mid m \in \mathbb{N}\},$$

where the multiplicity of roots from the first two subsets is 1 and from the third is 2, and the set of odd roots

$$\left\{(\pm\alpha/2, m) \mid m \in \tfrac{1}{2} + \mathbb{Z}_+\right\},$$

all having multiplicity 2.

In order to write down the determinant formula, let $\Lambda = \frac{j}{2}\alpha$, where $j \in \mathbb{C}$. The highest weight $(\Lambda, h) \equiv (j, h)$ of a Verma module over $N = 4$ superconformal algebra is determined by $h =$ the lowest eigenvalue of L_0 and $j =$ the eigenvalue of J_0^0 on the highest weight vector. We have:

$$h_{n,m}(k,j) = \frac{1}{4k}((mk - n)^2 - (k+1)^2 - j(j+2)),$$

$$\varphi_{n,m,\pm\alpha}(k,j) = mk + n \mp (j+1),$$

$$h_{m,\pm\alpha/2}(k,j) = \left(m^2 - \frac{1}{4}\right)k \mp m(j+1) - \frac{1}{2}.$$

Hence Remark 7.2 gives the following determinant formula (conjectured in [KeR]):

$$\begin{aligned}\det_{\hat\eta}(k,h,j) = &\prod_{m,n\in\mathbb{N}} (4kh-(mk-n)^2+(k+1)^2+j(j+2))^{P_W(\hat\eta-(0,mn))}\\ &\times \prod_{m,n\in\mathbb{N}} \varphi_{n,m,-\alpha}(k,j)^{P_W(\hat\eta-n(-\alpha,m))}\varphi_{n,m-1,\alpha}(k,j)^{P_W(\hat\eta-n(\alpha,m-1))}\\ &\times \prod_{m\in\frac12+\mathbb{Z}_+} \Big((h-h_{m,-\alpha/2}(k,j))^{2P_{W,(-\alpha/2,m)}(\hat\eta-(-\alpha/2,m))}\\ &\times (h-h_{m,\alpha/2}(k,j))^{2P_{W,(\alpha/2,m)}(\hat\eta-(\alpha/2,m))}\Big).\end{aligned}$$

8.5. $N=3$ superconformal algebra

As we shall see, the $N=3$ superconformal algebra is $W_k(\mathfrak{g},e_{-\theta})$, where $\mathfrak{g}=osp(3|2)$, tensored with one free fermion. The simple roots of $\mathfrak{g}$ are $\alpha_1,\ \alpha_2$, where α_1 is odd and α_2 is even, with scalar products:

$$(\alpha_1|\alpha_1)=0,\quad (\alpha_1|\alpha_2)=1/2,\quad (\alpha_2|\alpha_2)=-1/2,$$

with respect to the bilinear form $(a|b)=-\mathrm{str}\,ab$. The subalgebra $\mathfrak{g}^\natural\cong S\ell_2$ corresponds to the root α_2 so that its dual Coxeter number $h_0^\vee=-1/2$ and $\mathfrak{h}^\natural=\mathbb{C}\alpha_2$. Furthermore, $S_{1/2}=\{\alpha_1,\alpha_1+\alpha_2,\alpha_1+2\alpha_2\}$ consists of all odd positive roots of $\mathfrak{g}$. All positive even roots are α_2 and $\theta=2(\alpha_1+\alpha_2)$. For each positive root α we choose root vectors e_α and $e_{-\alpha}$ such that $(e_\alpha|e_{-\alpha})=1$.

We use the following notation:

$$h_{mn}=m\alpha_1+n\alpha_2,\quad e_{mn}=e_{h_{mn}},\quad f_{mn}=e_{-h_{mn}}.$$

We have: $\theta=h_{22}$, so that $x=h_{11}$ and $f=f_{22}$. We choose the positive root vectors such that the non-zero brackets are as follows:

$$[e_{10},e_{01}]=-e_{11},\quad [e_{10},e_{12}]=e_{22},\quad [e_{01},e_{11}]=e_{12},\quad [e_{11},e_{11}]=-e_{22}.$$

Then $\langle e_{10},e_{12}\rangle_{\mathrm{ne}}=1$, $\langle e_{11},e_{11}\rangle_{\mathrm{ne}}=-1$, hence all non-zero λ-brackets between the neutral fermions Φ_{10}, Φ_{12} and Φ_{11} are as follows:

$$[\Phi_{10\lambda}\Phi_{12}]=1,\quad [\Phi_{11\lambda}\Phi_{11}]=-1,\quad \text{and}\quad \Phi^{10}=\Phi_{12},\quad \Phi^{12}=\Phi_{10},\quad \Phi^{11}=-\Phi_{11}.$$

Since $h^\vee=1/2$ and $h_0^\vee=-1/2$, we have by (2.5):

$$[J^{(a)}{}_\lambda J^{(b)}]=J^{([a,b])}+\lambda(k+1)(a|b),\quad a,b\in\mathfrak{g}_0.$$

The d_0-closed fields, provided by Theorem 4.1, which strongly generate $W_k(\mathfrak{g}, e_{-\theta})$ are as follows:

$$J^0 = -4J^{(h_{01})} - 2 : \Phi_{10}\Phi_{12} :, \quad J^+ = -2J^{(e_{01})} - 2 : \Phi_{12}\Phi_{11} :,$$

$$J^- = 2J^{(f_{01})} - : \Phi_{10}\Phi_{11} :,$$

$$G^+ = J^{(f_{10})} + : \Phi_{12}J^{(h_{10})} : -\frac{1}{2} : \Phi_{11}J^{(e_{01})} : + (k+1)\partial\Phi_{12},$$

$$G^0 = J^{(f_{11})} - : \Phi_{12}J^{(f_{01})} : + \frac{1}{2} : \Phi_{10}J^{(e_{01})} : - : \Phi_{11}J^{(h_{11})} : - (k+1)\partial\Phi_{11},$$

$$G^- = J^{(f_{12})} + : \Phi_{10}J^{(h_{12})} : + : \Phi_{11}J^{(f_{01})} + (k+1)\partial\Phi_{10},$$

$$\begin{aligned} L = & -\frac{1}{k+1/2}(J^{(f)} + : \Phi_{12}J^{(f_{12})} : + : \Phi_{11}J^{(f_{11})} : + : \Phi_{10}J^{(f_{10})}) \\ & + \frac{1}{k+1/2}(: (J^{(h_{11})})^2 : - : (J^{(h_{01})})^2 : + \frac{1}{2} : J^{(e_{01})}J^{(f_{01})} : + \frac{1}{2} : J^{(f_{01})}J^{(e_{01})} : \\ & + (k+1)\partial J^{(h_{11})}) - (: \Phi_{12}\partial\Phi_{10} : + : \Phi_{10}\partial\Phi_{12} : - : \Phi_{11}\partial\Phi_{11} :). \end{aligned}$$

The field L is a Virasoro field with central charge

$$c = -6k - 7/2,$$

the fields $J^0,\ J^{\pm}$ (resp. $G^0,\ G^{\pm}$) being primary of conformal weight 1 (resp. 3/2). The remaining non-zero λ-brackets are as follows:

$$[J^0{}_\lambda J^{\pm}] = \pm 2J^{\pm}, \quad [J^0{}_\lambda J^0] = \frac{4\lambda}{3}\left(c + \frac{1}{2}\right), \quad [J^+{}_\lambda J^-] = J^0 + \frac{2\lambda}{3}\left(c + \frac{1}{2}\right),$$

$$[J^0{}_\lambda G^{\pm}] = \pm 2G^{\pm}, \quad [J^+{}_\lambda G^-] = -2G^0, \quad [J^-{}_\lambda G^+] = -G^0, \quad [J^+{}_\lambda G^0] = -2G^+,$$

$$[J^-{}_\lambda G^0] = -G^-,$$

$$[G^0{}_\lambda G^0] = \frac{1}{6}\left(c + \frac{1}{2}\right)L - \frac{1}{16} : J^0J^0 : + \frac{\lambda^2}{36}\left(c + \frac{1}{2}\right)(c-1), \quad [G^-{}_\lambda G^-] = -\frac{1}{4} : J^-J^- :,$$

$$[G^+{}_\lambda G^+] = -\frac{1}{16} : J^+J^+ :,$$

$$[G^{+}{}_{\lambda}G^{0}] = -\frac{1}{16} :J^{0}J^{+}: + \frac{1}{4}\left(\frac{1}{2} - \frac{1}{6}\left(c+\frac{1}{2}\right)\right)(\partial + 2\lambda)J^{+} - \frac{\lambda}{8}J^{+},$$

$$[G^{-}{}_{\lambda}G^{0}] = \frac{1}{8} :J^{0}J^{-}: + \frac{1}{2}\left(\frac{1}{2} - \frac{1}{6}\left(c+\frac{1}{2}\right)\right)(\partial + 2\lambda)J^{-} - \frac{\lambda}{4}J^{-},$$

$$[G^{+}{}_{\lambda}G^{-}] = -\frac{1}{6}\left(c+\frac{1}{2}\right)L + \frac{1}{8} :J^{-}J^{+}: + \frac{1}{4}\left(\frac{1}{2} - \frac{1}{6}\left(c+\frac{1}{2}\right)\right)(\partial + 2\lambda)J^{0}$$

$$-\frac{\lambda}{8}J^{0} - \frac{\lambda^{2}}{36}\left(c+\frac{1}{2}\right)(c-1).$$

In order to remove the quadratic terms in λ-brackets we introduce a free fermion Φ with the λ-bracket $[\Phi_{\lambda}\Phi] = -\left(k+\frac{1}{2}\right)$, and modify the fields L and G as follows:

$$\tilde{L} = L - \frac{1}{2k+1} :\partial\Phi\Phi:, \quad \tilde{G}^{+} = \frac{i}{\sqrt{k+1/2}}G^{+} - \frac{1}{4k+2} :J^{+}\Phi:,$$

$$\tilde{G}^{-} = \frac{-i}{\sqrt{k+1/2}}G^{-} - \frac{1}{2k+1} :J^{-}\Phi:, \quad \tilde{G}^{0} = \frac{-i}{\sqrt{k+1/2}}G^{0} + \frac{1}{4k+2} :J^{0}\Phi: .$$

Then $\tilde{L}$ is a Virasoro field with central charge

$$\tilde{c} = c + \tfrac{1}{2} = -6k - 3,$$

and we obtain the $N=3$ superconformal algebra, strongly generated by $\tilde{L}$, the primary fields $J^{\pm}$ and J^{0} of conformal weight 1, the primary fields $\tilde{G}^{\pm}$ and $\tilde{G}^{0}$ of conformal weight $3/2$ and the primary field Φ of conformal weight $1/2$. The remaining changed λ-brackets are as follows:

$$[J^{0}{}_{\lambda}\tilde{G}^{0}] = -2\lambda\Phi, \quad [J^{+}{}_{\lambda}\tilde{G}^{-}] = -2\tilde{G}^{0} + 2\lambda\Phi, \quad [J^{-}{}_{\lambda}\tilde{G}^{+}] = \tilde{G}^{0} + \lambda\Phi, \quad [\tilde{G}^{\pm}{}_{\lambda}\tilde{G}^{\pm}] = 0,$$

$$[\tilde{G}^{+}{}_{\lambda}\tilde{G}^{-}] = \tilde{L} + \frac{1}{4}(\partial + 2\lambda)J^{0} + \frac{\lambda^{2}}{6}\tilde{c}, \quad [\tilde{G}^{+}{}_{\lambda}\tilde{G}^{0}] = \frac{1}{4}(\partial + 2\lambda)J^{+}, \quad [\tilde{G}^{0}{}_{\lambda}\tilde{G}^{0}] = \tilde{L} + \frac{\lambda^{2}}{6}\tilde{c},$$

$$[\tilde{G}^{-}{}_{\lambda}\tilde{G}^{0}] = -\tfrac{1}{2}(\partial + 2\lambda)J^{-}, \quad [\tilde{G}^{+}{}_{\lambda}\Phi] = \tfrac{1}{4}J^{+}, \quad [\tilde{G}^{-}{}_{\lambda}\Phi] = \tfrac{1}{2}J^{-}, \quad [\tilde{G}^{0}{}_{\lambda}\Phi] = -\tfrac{1}{4}J^{0}.$$

We have: $\omega(J_{n}^{0}) = J_{-n}^{0}$, $\omega(J_{n}^{+}) = 2J_{-n}^{-}$, $\omega(G_{n}^{0}) = G_{-n}^{0}$, $\omega(G_{n}^{\pm}) = G_{-n}^{\mp}$, $\omega(\Phi_{n}) = \Phi_{-n}$.

The free field realization of the $N=3$ superconformal algebra is obtained from the formulas for $\tilde{L}$, J's and $\tilde{G}$'s by removing the terms containing $J^{(f)}$, $J^{(f_{10})}$, $J^{(f_{11})}$ and $J^{(f_{12})}$, and replacing $J^{(u)}$ by u for $u \in \mathfrak{g}_{0} = \mathfrak{h} + \mathbb{C}e_{01} + \mathbb{C}f_{01}$.

Let $\alpha = \alpha_2|_{\mathfrak{h}^\natural}$. Then $\Delta_+^\natural = \{\alpha\}$ and $\Delta' = \{\pm\alpha, 0\}$. The set of positive roots $\Delta_{N=3}^+$ of the $N = 3$ superconformal algebra consists of the set of even roots

$$\{(\pm\alpha, m) \mid m\in\mathbb{N}\} \cup \{(\alpha, 0)\} \cup \{(0, m) \mid m\in\mathbb{N}\},$$

where the multiplicity of a root from the first two subsets is 1 and from the third is 2, and the set of odd roots

$$\left\{(\pm\alpha, m) \mid m\in\tfrac{1}{2} + \mathbb{Z}_+\right\} \cup \left\{(0, m) \mid m\in\tfrac{1}{2} + \mathbb{Z}_+\right\},$$

where the multiplicity of a root from the first subset is 1 and from the second is 2 (due to the added free fermion Φ).

In order to write down the determinant formula, let $\Lambda = \frac{j}{2}\alpha$, where $j\in\mathbb{C}$. The highest weight $(\Lambda, h) \equiv (j, h)$ of the Verma module over $N = 3$ superconformal algebra is determined by $h =$ the lowest eigenvalue of L_0 and $j =$ the eigenvalue of J_0^0 on the highest weight vector. We have:

$$h_{n,2m}(k,j) = \frac{1}{2(2k+1)}\left((m(2k+1) - n)^2 - (k+1)^2 - \frac{1}{4}j(j+2)\right),$$

$$\varphi_{n,m,\pm\alpha}(k,j) = \tfrac{1}{4}(2m(2k+1) + n \mp (j+1)),$$

$$h_{m,\pm\alpha}(k,j) = \left(m^2 - \tfrac{1}{4}\right)k + \tfrac{1}{2}\left(m^2 - \tfrac{3}{4} \mp m(j+1)\right).$$

Hence Remark 7.2 gives the following determinant formula (cf. [M]):

$$\begin{aligned}
\det{}_{\hat\eta}(k,h,j) = {}& \left(k + \frac{1}{2}\right)^{P_{N=3,(0,1/2)}(\hat\eta-(0,1/2))} \\
& \times \prod_{\substack{m,n\in\frac{1}{2}\mathbb{N} \\ m-n\in\mathbb{Z}}} (h - h_{n,2m}(k,j))^{P_{N=3}(\hat\eta-(0,2mn))} \\
& \times \prod_{m,n\in\mathbb{N}} \varphi_{n,m,-\alpha}(k,j)^{P_{N=3}(\hat\eta-n(-\alpha,m))} \varphi_{n,m-1,\alpha}(k,j)^{P_{N=3}(\hat\eta-n(\alpha,m-1))} \\
& \times \prod_{m\in\frac{1}{2}+\mathbb{Z}_+} (h - h_{m,-\alpha}(k,j))^{P_{N=3,(-\alpha,m)}(\hat\eta-(-\alpha,m))} (h - h_{m,\alpha}(k,j))^{P_{N=3,(\alpha,m)}(\hat\eta-(\alpha,m))}
\end{aligned}$$

8.6. Big $N = 4$ superconformal algebra

In this subsection, $\mathfrak{g} = D(2,1;a)$, where $a\in\mathbb{C}\backslash\{-1,0\}$, is the family of exceptional 17-dimensional Lie superalgebras. The big $N = 4$ superconformal algebra (see [KL,S,STP]) is obtained from $W_k(\mathfrak{g}, e_{-\theta})$ by tensoring it with four free fermions and a free boson (cf. [GS]). Recall that $\mathfrak{g}$ is a contragredient Lie superalgebra with the following matrix of scalar products of simple roots α_i $(i = 1, 2, 3)$ [K1]:

$$((\alpha_i|\alpha_j))_{i,j=1}^3 = -\frac{1}{a+1}\begin{pmatrix} 2 & -1 & 0 \\ -1 & 0 & -a \\ 0 & -a & 2a \end{pmatrix}.$$

As in Section 8.5, we use the following notation:

$$h_{mnp} = m\alpha_1 + n\alpha_2 + p\alpha_3, \quad e_{mnp} = e_{h_{mnp}}, \quad f_{mnp} = e_{-h_{mnp}}.$$

The set of even (resp. odd) positive roots is:

$$h_{100}, h_{001}, h_{121} \text{ (resp. } h_{010}, h_{110}, h_{011}, h_{111}).$$

The highest root $\theta = h_{121}$, and the matrix of scalar products is normalized in such a way that $(\theta|\theta) = 2$.

We choose positive root vectors e_α such that all non-zero brackets (up to the order) between them are as follows: $[e_{100}, e_{010}] = e_{110}$, $[e_{100}, e_{011}] = e_{111}$, $[e_{010}, e_{001}] = e_{011}$, $[e_{010}, e_{111}] = e_{121}$, $[e_{001}, e_{110}] = -e_{111}$, $[e_{110}, e_{011}] = -e_{121}$, and we choose negative root vectors f_α such that $[e_\alpha, f_\alpha] = \alpha$.

For the quantum reduction we take the following $s\ell_2$-triple: $f = f_{121}$, $e = \frac{1}{2}e_{121}$, $x = \frac{1}{2}h_{121}$, so that $\mathfrak{h}^\natural = \mathbb{C}\alpha_1 + \mathbb{C}\alpha_3$, and $\mathfrak{g}^\natural = \mathfrak{h}^\natural + \mathbb{C}e_{\alpha_1} + \mathbb{C}f_{\alpha_1} + \mathbb{C}e_{\alpha_3} + \mathbb{C}f_{\alpha_3}$ is isomorphic to $s\ell_2 \oplus s\ell_2$. The subspace $\mathfrak{g}_{1/2}$ is spanned by four odd elements: e_{010}, e_{111}, e_{110}, e_{011}, and all non-zero values of the (symmetric) bilinear form $\langle .,. \rangle_{\mathrm{ne}}$ on $\mathfrak{g}_{1/2}$ are:

$$\langle e_{010}, e_{111} \rangle_{\mathrm{ne}} = 1, \quad \langle e_{110}, e_{011} \rangle_{\mathrm{ne}} = -1.$$

Hence the corresponding free neutral fermions satisfy the following non-zero λ-brackets:

$$[\Phi_{010\,\lambda}\Phi_{111}] = 1, \quad [\Phi_{110\,\lambda}\Phi_{011}] = -1,$$

and

$$\Phi^{010} = \Phi_{111}, \quad \Phi^{111} = \Phi_{010}, \quad \Phi^{110} = -\Phi_{011}, \quad \Phi^{011} = -\Phi_{110}.$$

The d_0-closed fields, provided by Theorem 4.1, which strongly generate $W_k(\mathfrak{g}, e_{-\theta})$ are as follows:

$$J^0 = -(a+1)J^{(h_{100})} - (:\Phi_{010}\Phi_{111}: - :\Phi_{110}\Phi_{011}:),$$

$$J'^0 = -\frac{a+1}{a}J^{(h_{001})} - (:\Phi_{010}\Phi_{111}: - :\Phi_{110}\Phi_{011}:),$$

$$J^+ = J^{(e_{100})} + :\Phi_{110}\Phi_{111}:, \quad J'^+ = J^{(e_{001})} - :\Phi_{011}\Phi_{111}:,$$

$$J^- = -(a+1)J^{(f_{100})} - :\Phi_{010}\Phi_{011}:,\quad J'^- = -\frac{a+1}{a}J^{(f_{001})} + :\Phi_{010}\Phi_{110}:,$$

$$G^{++} = J^{(f_{010})} + :\Phi_{111}J^{(h_{010})}: + \frac{1}{a+1}:\Phi_{011}J^{(e_{100})}: - \frac{a}{a+1}:\Phi_{110}J^{(e_{001})}:$$
$$+ \frac{a-1}{a+1}:\Phi_{110}\Phi_{011}\Phi_{111}: + (k+1)\partial\Phi_{111},$$

$$G^{--} = J^{(f_{111})} + :\Phi_{010}J^{(h_{111})}: - :\Phi_{110}J^{(f_{100})}: + :\Phi_{011}J^{(f_{001})}:$$
$$- \frac{a-1}{a+1}:\Phi_{010}\Phi_{110}\Phi_{011}: + (k+1)\partial\Phi_{010},$$

$$G^{-+} = J^{(f_{110})} - :\Phi_{011}J^{(h_{110})}: + :\Phi_{111}J^{(f_{100})}: + \frac{a}{a+1}:\Phi_{010}J^{(e_{001})}:$$
$$- \frac{a-1}{a+1}:\Phi_{010}\Phi_{011}\Phi_{111}: - (k+1)\partial\Phi_{011},$$

$$G^{+-} = J^{(f_{011})} - :\Phi_{110}J^{(h_{011})}: - \frac{1}{a+1}:\Phi_{010}J^{(e_{100})}: - :\Phi_{111}J^{(f_{001})}:$$
$$+ \frac{a-1}{a+1}:\Phi_{010}\Phi_{110}\Phi_{111}: - (k+1)\partial\Phi_{110},$$

$$\begin{aligned}
L = {} & -\frac{1}{k}J^{(f)} - \frac{1}{k}\sum_{\alpha\in S_{1/2}} :\Phi_\alpha J^{(f_\alpha)}: \\
& + \frac{1}{4k}\left(:(J^{(x)})^2: - (a+1):(J^{(h_{100})})^2: - \frac{a+1}{a}:(J^{(h_{001})})^2:\right) + \frac{k+1}{k}\partial J^{(x)} \\
& + \frac{1}{2k}(:J^{(e_{100})}J^{(f_{100})}: + :J^{(f_{100})}J^{(e_{100})}: + :J^{(e_{001})}J^{(f_{001})}: + :J^{(f_{001})}J^{(e_{001})}:) \\
& - \frac{1}{2}(:\Phi_{111}\partial\Phi_{010}: + :\Phi_{010}\partial\Phi_{111}: - :\Phi_{110}\partial\Phi_{011}: - :\Phi_{011}\partial\Phi_{110}:).
\end{aligned}$$

The fields J's (resp. G's) are primary with respect to L of conformal weight 1 (resp. 3/2). The non-zero λ-brackets between these fields are as follows:

$$[J^0{}_\lambda J^0] = -2\lambda((a+1)k+1),\quad [J'^0{}_\lambda J'^0] = -2\lambda\left(\frac{a+1}{a}k+1\right),$$

$$[J^+{}_\lambda J^-] = J^0 - \lambda((a+1)k+1),\quad [J'^+{}_\lambda J'^-] = J'^0 - \lambda\left(\frac{a+1}{a}k+1\right),$$

$$[J^0{}_\lambda J^\pm] = \pm 2J^\pm,\quad [J'^0{}_\lambda J'^\pm] = \pm 2J'^\pm,$$

$$[J^0{}_\lambda G^{\pm\pm}] = \pm G^{\pm\pm},\quad [J^0{}_\lambda G^{\pm\mp}] = \pm G^{\pm\mp},\quad [J'^0{}_\lambda G^{\pm\pm}] = \pm G^{\pm\pm},$$

$$[J'^0{}_\lambda G^{\pm\mp}] = \mp G^{\pm\mp},$$

$$[J^+{}_\lambda G^{--}] = -G^{+-}, \quad [J^+{}_\lambda G^{-+}] = -G^{++}, \quad [J'^+{}_\lambda G^{--}] = G^{-+},$$

$$[J'^+{}_\lambda G^{+-}] = G^{++},$$

$$[J^-{}_\lambda G^{++}] = -G^{-+}, \quad [J^-{}_\lambda G^{+-}] = -G^{--}, \quad [J'^-{}_\lambda G^{++}] = G^{+-},$$

$$[J'^-{}_\lambda G^{-+}] = G^{--},$$

$$[G^{++}{}_\lambda G^{++}] = \frac{2a}{(a+1)^2} :J^+J^+:, \quad [G^{--}{}_\lambda G^{--}] = \frac{2}{(a+1)^2} :J^-J^-:,$$

$$[G^{-+}{}_\lambda G^{-+}] = -\frac{2a}{(a+1)^2} :J^-J'^+:, \quad [G^{+-}{}_\lambda G^{+-}] = -\frac{2a}{(a+1)^2} :J^+J'^-:,$$

$$\begin{aligned}[G^{++}{}_\lambda G^{--}] = {} & kL + \frac{1}{4}\left(\frac{1}{a+1} :J^0J^0: + \frac{a}{a+1} :J'^0J'^0: - \frac{1}{(a+1)^2} :(J^0 + aJ'^0)^2:\right) \\ & + \frac{a}{(a+1)^2}(:J^+J^-: + :J'^+J'^-:) - \frac{1}{2(a+1)}\partial(J^0 + aJ'^0) \\ & + \frac{k+1}{2(a+1)}(\partial + 2\lambda)(J^0 + aJ'^0) - \frac{\lambda}{(a+1)^2}(J^0 + a^2J'^0) \\ & - \lambda^2\left(k(k+1) + \frac{a}{(a+1)^2}\right),\end{aligned}$$

$$\begin{aligned}[G^{-+}{}_\lambda G^{+-}] = {} & -kL + \frac{1}{4}\left(-\frac{1}{a+1} :J^0J^0: - \frac{a}{a+1} :J'^0J'^0: + \frac{1}{(a+1)^2} :(J^0 - aJ'^0)^2:\right) \\ & - \frac{a}{(a+1)^2}(:J^+J^-: + :J'^+J'^-:) + \frac{1}{2(a+1)}\partial(J^0 + aJ'^0) \\ & - \frac{1}{(a+1)^2}\partial J^0 + \frac{k+1}{2(a+1)}(\partial + 2\lambda)(J^0 - aJ'^0) \\ & - \frac{\lambda}{(a+1)^2}(J^0 - a^2J'^0) + \lambda^2\left(k(k+1) + \frac{a}{(a+1)^2}\right),\end{aligned}$$

$$[G^{++}{}_\lambda G^{-+}] = \frac{a}{(a+1)^2} :J^0J'^+: + \frac{a}{a+1}\left(\frac{a}{a+1} + k + 1\right)(\partial + 2\lambda)J'^+,$$

$$[G^{++}{}_\lambda G^{+-}] = -\frac{a}{(a+1)^2} :J'^0J^+: - \frac{1}{a+1}\left(k + 1 + \frac{1}{a+1}\right)(\partial + 2\lambda)J^+,$$

$$[G^{--}{}_{\lambda}G^{-+}] = \frac{a}{(a+1)^2} :J'^0 J^-: + \frac{1}{a+1}\left(k+1-\frac{1}{a+1}\right)(\partial+2\lambda)J^-,$$

$$[G^{--}{}_{\lambda}G^{+-}] = -\frac{a}{(a+1)^2} :J^0 J'^-: - \frac{a}{a+1}\left(k+1-\frac{a}{a+1}\right)(\partial+2\lambda)J'^-.$$

Since $h^\vee = 0$ and $\mathrm{sdim}\,\mathfrak{g} = 1$, by (5.7), the Virasoro central charge is equal:

$$c = -6k - 3.$$

The big $N=4$ superconformal algebra is obtained from $W_k(D(2,1;a), e_{-\theta})$ by tensoring the latter vertex algebra by the vertex algebra strongly generated by four (odd) free fermions $\sigma^{--},\ \sigma^{++},\ \sigma^{+-},\ \sigma^{-+}$ with non-zero λ-brackets $[\sigma^{--}{}_\lambda\sigma^{++}]=k$, $[\sigma^{+-}{}_\lambda\sigma^{-+}]=k$, and one (even) free boson ξ with λ-bracket $[\xi_\lambda\xi]=\lambda k$.

Then the five fields $\sigma^{--},\ \sigma^{++},\ \sigma^{+-},\ \sigma^{-+},\ \xi$ along with the following modifications of the six fields $J^0,\ J'^0,\ J^\pm,\ J'^\pm$, the four fields $G^{++},\ G^{--},\ G^{+-},\ G^{-+}$ and the field L, close in the big $N=4$ superconformal algebra with Virasoro central charge $\tilde{c} = -6k$ (cf. [IKL]):

$$\tilde{J}^0 = J^0 - \frac{1}{k}:\sigma^{--}\sigma^{++}: + \frac{1}{k}:\sigma^{+-}\sigma^{-+}:, \quad \tilde{J}'^0 = J'^0 + \frac{1}{k}:\sigma^{--}\sigma^{++}: + \frac{1}{k}:\sigma^{+-}\sigma^{-+}:,$$

$$\tilde{J}^+ = J^+ + \frac{a}{k}:\sigma^{+-}\sigma^{++}:, \quad \tilde{J}^- = J^- + \frac{1}{ak}:\sigma^{--}\sigma^{-+}:,$$

$$\tilde{J}'^+ = J'^+ + \frac{1}{k}:\sigma^{-+}\sigma^{++}:, \quad \tilde{J}'^- = J'^- + \frac{1}{k}:\sigma^{--}\sigma^{+-}:,$$

$$\begin{aligned}\tilde{G}^{++} = {} & \frac{1}{\sqrt{k}}G^{++} - \frac{1}{k(a+1)}:J^+\sigma^{-+}: + \frac{a}{k(a+1)}:J'^+\sigma^{+-}: + \frac{a}{2k(a+1)}:J^0\sigma^{++}: \\ & - \frac{a}{2k(a+1)}:J'^0\sigma^{++}: + \frac{1}{k}\left(\frac{a}{2}\right)^{1/2}:\xi\sigma^{++}: + \frac{a}{k^2(a+1)}:\sigma^{++}\sigma^{+-}\sigma^{-+}:,\end{aligned}$$

$$\begin{aligned}\tilde{G}^{--} = {} & \frac{1}{\sqrt{k}}G^{--} + \frac{a}{k(a+1)}:J^-\sigma^{+-}: - \frac{1}{k(a+1)}:J'^-\sigma^{-+}: - \frac{1}{2k(a+1)}:J^0\sigma^{--}: \\ & + \frac{1}{2k(a+1)}:J'^0\sigma^{--}: + \frac{1}{k}\left(\frac{1}{2a}\right)^{1/2}:\xi\sigma^{--}: - \frac{1}{k^2(a+1)}:\sigma^{--}\sigma^{+-}\sigma^{-+}:,\end{aligned}$$

$$\begin{aligned}\tilde{G}^{+-} = {} & -\frac{1}{\sqrt{k}}G^{+-} + \frac{1}{k(a+1)}:J^+\sigma^{--}: + \frac{a}{k(a+1)}:J'^-\sigma^{++}: + \frac{a}{2k(a+1)}:J^0\sigma^{+-}: \\ & + \frac{a}{2k(a+1)}:J'^0\sigma^{+-}: + \frac{1}{k}\left(\frac{a}{2}\right)^{1/2}:\xi\sigma^{+-}: - \frac{a}{k^2(a+1)}:\sigma^{+-}\sigma^{--}\sigma^{++}:,\end{aligned}$$

$$\tilde{G}^{-+} = \frac{1}{\sqrt{k}} G^{-+} - \frac{1}{k(a+1)} : J'^{+}\sigma^{--} : - \frac{a}{k(a+1)} : J^{-}\sigma^{++} : - \frac{1}{2k(a+1)} : J^{0}\sigma^{-+} :$$
$$- \frac{1}{2k(a+1)} : J'^{0}\sigma^{-+} : + \frac{1}{k}\left(\frac{1}{2a}\right)^{1/2} : \xi\sigma^{-+} : + \frac{1}{k^2(a+1)} : \sigma^{-+}\sigma^{--}\sigma^{++} :,$$

$$\tilde{L} = L + \frac{1}{2k}(: \partial\sigma^{--}\sigma^{++} : + : \partial\sigma^{++}\sigma^{--} : + : \partial\sigma^{-+}\sigma^{+-} : + : \partial\sigma^{+-}\sigma^{-+} : + : \xi^2 :).$$

The fields σ's, ξ, $\tilde{J}$'s and $\tilde{G}$'s are primary with respect to $\tilde{L}$ of conformal weight 1/2, 1, 1 and 3/2, respectively. The λ-brackets for the pairs $(\tilde{J}, \tilde{J}')$ are zero, and the non-zero λ-brackets for the pairs $(\tilde{J}, \tilde{G})$, $(\tilde{J}', \tilde{G})$, $(\tilde{J}, \tilde{J})$, and $(\tilde{J}', \tilde{J}')$ are as follows:

$$[\tilde{J}^0{}_\lambda \tilde{G}^{++}] = \tilde{G}^{++} - \lambda a\sigma^{++}, \quad [\tilde{J}^0{}_\lambda \tilde{G}^{--}] = -\tilde{G}^{--} + \lambda\sigma^{--}, \quad [\tilde{J}^0{}_\lambda \tilde{G}^{-+}] = -\tilde{G}^{-+} + \lambda\sigma^{-+},$$

$$[\tilde{J}^0{}_\lambda \tilde{G}^{+-}] = \tilde{G}^{+-} - \lambda a\sigma^{+-}, \quad [\tilde{J}^+_\lambda \tilde{G}^{--}] = \tilde{G}^{+-} - \lambda a\sigma^{+-}, \quad [\tilde{J}^+{}_\lambda \tilde{G}^{-+}] = -\tilde{G}^{++} + \lambda a\sigma^{++},$$

$$[\tilde{J}^-{}_\lambda \tilde{G}^{++}] = -\tilde{G}^{-+} + \lambda\sigma^{-+}, \quad [\tilde{J}^-{}_\lambda \tilde{G}^{+-}] = \tilde{G}^{--} - \lambda\sigma^{--},$$

$$[\tilde{J}'^0{}_\lambda \tilde{G}^{++}] = \tilde{G}^{++} + \lambda\sigma^{++}, \quad [\tilde{J}'^0{}_\lambda \tilde{G}^{--}] = -\tilde{G}^{--} - \frac{\lambda}{a}\sigma^{--}, \quad [\tilde{J}'^0{}_\lambda \tilde{G}^{-+}] = \tilde{G}^{-+} + \frac{\lambda}{a}\sigma^{-+},$$

$$[\tilde{J}'^0{}_\lambda \tilde{G}^{+-}] = -\tilde{G}^{+-} - \lambda\sigma^{+-}, \quad [\tilde{J}'^+{}_\lambda \tilde{G}^{--}] = \tilde{G}^{-+} + \frac{\lambda}{a}\sigma^{-+}, \quad [\tilde{J}'^+{}_\lambda \tilde{G}^{+-}] = -\tilde{G}^{++} - \lambda\sigma^{++},$$

$$[\tilde{J}'^-{}_\lambda \tilde{G}^{++}] = -\tilde{G}^{+-} - \lambda\sigma^{+-}, \quad [\tilde{J}'^-{}_\lambda \tilde{G}^{--}] = \tilde{G}^{--} + \frac{\lambda}{a}\sigma^{--},$$

$$[\tilde{J}^0{}_\lambda \tilde{J}^0] = \lambda\frac{\tilde{c}(a+1)}{3}, \quad [\tilde{J}^0{}_\lambda \tilde{J}^\pm] = \pm 2\tilde{J}^\pm, \quad [\tilde{J}^+{}_\lambda \tilde{J}^-] = \tilde{J}^0 + \lambda\frac{\tilde{c}(a+1)}{6},$$

$$[\tilde{J}'^0{}_\lambda \tilde{J}'^0] = \lambda\frac{\tilde{c}(a+1)}{3a},$$

$$[\tilde{J}'^0{}_\lambda \tilde{J}'^\pm] = \pm 2\tilde{J}'^\pm, \quad [\tilde{J}'^+{}_\lambda \tilde{J}'^-] = \tilde{J}'^0 + \lambda\frac{\tilde{c}(a+1)}{6a}.$$

The non-zero λ-brackets for the pairs $(\tilde{J}, \sigma)$ are as follows:

$$[\tilde{J}^0{}_\lambda \sigma^{\pm\pm}] = \pm\sigma^{\pm\pm}, \quad [\tilde{J}'^0{}_\lambda \sigma^{\pm\mp}] = \pm\sigma^{\pm\mp},$$

$$[\tilde{J}^+{}_\lambda \sigma^{=\mp}] = \pm a\sigma^{\overset{+}{+}\mp}, \quad [\tilde{J}^-{}_\lambda \sigma^{\overset{+}{+}\pm}] = \mp\frac{1}{a}\sigma^{=\pm},$$

$$[\tilde{J}'^0{}_\lambda \sigma^{\pm\pm}] = \pm\sigma^{\pm\pm}, \quad [\tilde{J}'^0{}_\lambda \sigma^{\pm\mp}] = \mp\sigma^{\pm\mp},$$

$$[\tilde{J}'^+{}_\lambda \sigma^{\mp=}] = \pm\sigma^{\mp\ddagger}, \quad [\tilde{J}'^-{}_\lambda \sigma^{\pm\ddagger}] = \mp\sigma^{\pm=}.$$

The non-zero λ-brackets for the pairs $(\tilde{G},\sigma)$ and $(\tilde{G},\xi)$ are as follows:

$$[\tilde{G}^{\ddagger\pm}{}_\lambda \sigma^{=\mp}] = \frac{a}{2(a+1)}(\tilde{J}^0 \mp \tilde{J}'^0) + \left(\frac{a}{2}\right)^{1/2}\xi,$$

$$[\tilde{G}^{\ddagger\pm}{}_\lambda \sigma^{=\pm}] = \frac{a}{a+1}\tilde{J}'^\pm, \quad [\tilde{G}^{=\mp}{}_\lambda \sigma^{\ddagger\mp}] = \pm\frac{1}{2(a+1)}(\tilde{J}'^0 \mp \tilde{J}^0) + \left(\frac{1}{2a}\right)^{1/2}\xi,$$

$$[\tilde{G}^{=\mp}{}_\lambda \sigma^{\ddagger\mp}] = -\frac{1}{a+1}\tilde{J}'^\mp, \quad [\tilde{G}^{=\mp}{}_\lambda \sigma^{=\pm}] = \pm\frac{a}{a+1}\tilde{J}^-, \quad [\tilde{G}^{\ddagger\mp}{}_\lambda \sigma^{\ddagger\pm}] = -\frac{1}{a+1}\tilde{J}^+,$$

$$[\tilde{G}^{\ddagger\pm}{}_\lambda \xi] = (\partial+\lambda)\left(\frac{a}{2}\right)^{1/2}\sigma^{\ddagger\pm}, \quad [\tilde{G}^{=\mp}{}_\lambda \xi] = (\partial+\lambda)\left(\frac{1}{2a}\right)^{1/2}\sigma^{=\mp}.$$

Finally, the non-zero λ-brackets between the $\tilde{G}$'s are as follows:

$$[\tilde{G}^{\pm\ddagger}{}_\lambda \tilde{G}^{\mp=}] = \tilde{L} + \frac{1}{(a+1)}(\partial+2\lambda)(\pm\tilde{J}^0 + a\tilde{J}'^0) + \frac{\lambda^2}{6}\tilde{c},$$

$$[\tilde{G}^{\pm\pm}{}_\lambda \tilde{G}^{\mp\pm}] = \mp\frac{a}{a+1}(\partial+2\lambda)\tilde{J}'^\pm, \quad [\tilde{G}^{\pm\pm}{}_\lambda \tilde{G}^{\pm\mp}] = \mp\frac{1}{a+1}(\partial+2\lambda)\tilde{J}^\pm.$$

We have: $\omega(J_n^0) = J_{-n}^0$, $\omega(J_n^\pm) = J_{-n}^\mp$, $\omega(J_n'^0) = J_{-n}'^0$, $\omega(J_n'^\pm) = J_{-n}'^\mp$, $\omega(G_n^{++}) = G_{-n}^{--}$, $\omega(G_n^{+-}) = G_{-n}^{-+}$, $\omega(\sigma_n^{--}) = -a\sigma_{-n}^{++}$, $\omega(\sigma_n^{-+}) = -a\sigma_{-n}^{+-}$, $\omega(\xi_n) = -\xi_{-n}$.

Let $\alpha = \alpha_1|_{\mathfrak{h}^\natural}$ and $\alpha' = \alpha_3|_{\mathfrak{h}^\natural}$. Then $\Delta_+^\natural = \{\alpha,\alpha'\}$ and $\Delta' = \{\pm\frac{1}{2}(\alpha+\alpha'), \pm\frac{1}{2}(\alpha-\alpha')\}$. The set of positive roots Δ_W^+ consists of the set of even roots:

$$\{(\pm\alpha,m),(\pm\alpha',m)\,|\,m\in\mathbb{N}\}\cup\{(\alpha,0),(\alpha',0)\}\cup(0,m)\,|\,m\in\mathbb{N}\},$$

where the multiplicity of a root from the first two subsets is 1 and from the third is 3, and the set of odd roots:

$$\{(\pm\tfrac{1}{2}(\alpha+\alpha'),m),(\pm\tfrac{1}{2}(\alpha-\alpha'),m)\,|\,m\in\tfrac{1}{2}+\mathbb{Z}_+\},$$

all of multiplicity 1.

In order to write down the determinant formula, let $\Lambda = \frac{1}{2}(j\alpha + j'\alpha')$. The highest weight (Λ,h) of the Verma module over $W_k(D(2,1;a),e_{-\theta})$ is determined by the triple (j,j',h), where $h=$ the lowest eigenvalue of L_0 and j ad j' are the eigenvalues of

J_0^0 and $J_0^{\prime 0}$ on the highest weight vector. We have:

$$h_{n,m}(k,j,j') = \frac{1}{4k}\left((mk-n)^2 - (k+1)^2 - \frac{(j+1)^2 + a(j'+1)^2}{a+1} + 1\right),$$

$$\varphi_{n,m,\pm\alpha}(k,j,j') = \frac{1}{a+1}((a+1)mk + n \mp (j+1)),$$

$$\varphi_{n,m,\pm\alpha'}(k,j,j') = \frac{1}{a+1}((a+1)mk + an \mp a(j'+1)),$$

$$h_{m,\pm\frac{1}{2}(\alpha+\alpha')}(k,j,j') = \frac{-a(j-j')^2}{4(a+1)^2 k} + \left(m^2 - \frac{1}{4}\right)k \mp \frac{m(j+1+a(j'+1))}{a+1} - \frac{1}{2},$$

$$h_{m,\pm\frac{1}{2}(\alpha-\alpha')}(k,j,j') = \frac{-a(j+j'+2)^2}{4(a+1)^2 k} + \left(m^2 - \frac{1}{4}\right)k \mp \frac{m(j+1-a(j'+1))}{a+1} - \frac{1}{2}.$$

Hence Remark 7.2 gives the following determinant formula for $W_k(D(2,1;a), e_{-\theta})$:

$$\begin{aligned} \det_{\hat{\eta}}(k,h,j,j') &= k^{2\sum_{m,n\in\mathbb{N}} P_W(\hat{\eta}-(0,mn))} \prod_{m,n\in\mathbb{N}} (h - h_{n,m}(k,j,j'))^{P_W(\hat{\eta}-(0,mn))} \\ &\times \prod_{\substack{m,n\in\mathbb{N} \\ \beta=\alpha,\alpha'}} \varphi_{n,m,-\beta}(k,j,j')^{P_W(\hat{\eta}-n(-\beta,m))} \varphi_{n,m-1,\beta}(k,j,j')^{P_W(\hat{\eta}-n(\beta,m-1))} \\ &\times \prod_{\substack{m\in\frac{1}{2}+\mathbb{Z}_+ \\ \gamma=\pm\frac{1}{2}(\alpha+\alpha'),\pm\frac{1}{2}(\alpha-\alpha')}} (h - h_{m,\gamma}(k,j,j'))^{P_{W,(\gamma,m)}(\hat{\eta}-(\gamma,m))}. \end{aligned}$$

The determinant formula for the big $N=4$ superconformal algebra is obtained by a simple modification of the above formula. First the multiplicities of the following roots increase by 1: $(0,m)$ (due to the added free boson) and $\left(\pm\frac{1}{2}(\alpha+\alpha'),m\right)$, $\left(\pm\frac{1}{2}(\alpha-\alpha'),m\right)$ (due to the added free fermions). This leads to the obvious changes of exponents. Second, one should add the obvious power of the eigenvalue of the zero's mode of the added free boson ξ.

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